A MATHEMATICAL-COMPUTER SIMULATION OF THE DYNAMICS OF A FREIGHT ELEMENT IN A RAILROAD FREIGHT CAR

JULY 1977
SUMMARY REPORT

This document is available to the public through
The National Technical Information Service,
Springfield, Virginia 22161

Prepared for
U.S. DEPARTMENT OF TRANSPORTATION
FEDERAL RAILROAD ADMINISTRATION
Office of Research and Development
Washington, D.C. 20590
NOTICE

The United States Government does not endorse products or manufacturers. Trade or manufacturers' names appear herein solely because they are considered essential to the object of this report.
**Abstract**

This research studies the dynamic response of a freight element, inside a typical freight box car under service conditions, by a computer-model simulation technique. A 27 degree of freedom mathematical model has been developed to represent the freight car, truck and freight element, with the car body as a single rigid mass. This model has been validated against published railroad research data. This model is a more detailed one than most previously published simulations, and has additional characteristics. One is the option of modeling dry friction dampers by either Coulomb friction or equivalent viscous damping. A second improvement is the facility to express the response of the system in either time or frequency domain. The computer simulation shows that the critical roll mode speed of a representative 70-ton box car is around 17.5 mph. The maximum car body roll angle is 11.4° peak to peak, the maximum wheel load is 69,000 lb/wheel, and wheel lift durations are 0.2-0.4 sec. For a specific freight element near the roof maximum lateral accelerations of 1.5 g peak to peak at 0.64 Hz were calculated. At 50 mph, this value becomes 0.2 g at 2 Hz. Vertical acceleration of 0.1 g at 1.25 Hz is computed for freight near the car body center of gravity at 50 mph. The mathematical model can be used for parametric studies on designs of the car body and truck.

Cushioning requirements for freight/package systems subjected to vibrations inside a freight car can also be established.
PREFACE

The work reported here is part of a Research Program into various technical problems in Rail Transportation which is currently underway at Illinois Inst. of Tech., Chicago, Ill. The Program is under the joint sponsorship of the U.S. Department of Transportation, Association of American Railroads, General Motors Corp. (Electro-Motive Division), and Illinois Institute of Technology under Contract Number DOT-OS-40103.

The Program of work is being conducted in the Department of Mechanics, Mechanical and Aerospace Engineering at IIT, with Dr. Sudhir Kumar, Department Chairman, as Project Director. Three technical areas of Rail Transportation are being investigated in this research program, and these are:

1) Freight Damage
2) Wheel/rail Friction
3) Diesel Engine Noise

This report is a Technical Report on a part of the work completed on Freight Damage in this project.

The support of the sponsors is gratefully acknowledged, and we are indebted to a number of individuals for their assistance in the course of this work. Specifically, our gratitude is due to Dr. S. Kumar, Project Director, who has provided invaluable advice and assistance in this work, and to Messrs. L. Olson and T. Tse, of AAR, for their continued help and advice. Also, we appreciate the effort put into the project by Mr. S. Shah of IIT, during the summer of 1974. Gratitude is also due to Mr. R. Bullock, Senior Project Engineer of Standard Car Truck Co. of Chicago, for his valuable advice and support for this project.

K. S.
T. W.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>PREFACE</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF ABBREVIATIONS AND SYMBOLS</td>
<td>ix</td>
</tr>
<tr>
<td>1. INTRODUCTION TO FREIGHT DAMAGE ASPECTS</td>
<td>1</td>
</tr>
<tr>
<td>2. SELECTION OF A BOX CAR AND DETERMINATION OF ITS CHARACTERISTICS</td>
<td>5</td>
</tr>
<tr>
<td>2.1 The Car Body</td>
<td>5</td>
</tr>
<tr>
<td>2.2 The Center Plates</td>
<td>6</td>
</tr>
<tr>
<td>2.3 The Side Bearings</td>
<td>6</td>
</tr>
<tr>
<td>2.4 The Center Plate Extension Pads</td>
<td>6</td>
</tr>
<tr>
<td>2.5 The Axle, Wheel Set and Sideframes</td>
<td>8</td>
</tr>
<tr>
<td>2.6 Suspension Springs</td>
<td>8</td>
</tr>
<tr>
<td>2.7 The Friction Shoe and Spring-Steel Wear Plate</td>
<td>8</td>
</tr>
<tr>
<td>2.8 Rail Profile and Subgrade Structures</td>
<td>9</td>
</tr>
<tr>
<td>2.9 Couplers</td>
<td>9</td>
</tr>
<tr>
<td>3. THE MATHEMATICAL MODEL</td>
<td>11</td>
</tr>
<tr>
<td>3.1 General Description</td>
<td>11</td>
</tr>
<tr>
<td>3.2 Car Body and Truck Bolsters Interface</td>
<td>11</td>
</tr>
<tr>
<td>3.3 Truck Bolsters and Sideframes Interface</td>
<td>11</td>
</tr>
<tr>
<td>3.4 Rail Profile and Subgrade</td>
<td>14</td>
</tr>
<tr>
<td>3.5 Coupler Forces</td>
<td>15</td>
</tr>
<tr>
<td>3.6 Car and Freight Element Interface</td>
<td>15</td>
</tr>
<tr>
<td>4. EQUATIONS OF MOTION AND METHOD OF SOLUTION</td>
<td>17</td>
</tr>
<tr>
<td>4.1 Equations of Motion</td>
<td>17</td>
</tr>
<tr>
<td>4.2 Solution by Computer Iterative Method</td>
<td>18</td>
</tr>
<tr>
<td>5. COMPUTER PROGRAM</td>
<td>19</td>
</tr>
<tr>
<td>5.1 MAIN Program</td>
<td>19</td>
</tr>
<tr>
<td>5.2 Subroutine DELGAP</td>
<td>19</td>
</tr>
<tr>
<td>5.3 Subroutine DELGIB</td>
<td>19</td>
</tr>
<tr>
<td>5.4 Subroutine SPRING</td>
<td>20</td>
</tr>
<tr>
<td>5.5 Subroutine ACCELN</td>
<td>20</td>
</tr>
<tr>
<td>5.6 Subroutine RUNG</td>
<td>20</td>
</tr>
<tr>
<td>5.7 Subroutine CPLATE</td>
<td>22</td>
</tr>
<tr>
<td>5.8 Subroutine CAL</td>
<td>22</td>
</tr>
<tr>
<td>Subroutine T5000</td>
<td>Page</td>
</tr>
<tr>
<td>------------------</td>
<td>------</td>
</tr>
<tr>
<td>Subroutine SGNFUN</td>
<td>22</td>
</tr>
<tr>
<td>Nonlinear Modeling</td>
<td>23</td>
</tr>
<tr>
<td>Computer Program for Frequency Analysis</td>
<td>23</td>
</tr>
</tbody>
</table>

6. COMPUTER MODEL VALIDATION | 25

6.1 Methods of Validation | 25
6.2 Comparison of IIT Model with Other Models | 26
6.3 Validation of IIT Model | 26

7. DYNAMICS OF A SINGLE FREIGHT ELEMENT | 35

7.1 Freight Dynamics Study by Computer Model Simulation | 35
7.2 Case Studies | 35

- Case 1. Rocking Mode, Freight Element near Roof of Box Car - 17.5 mph
- Case 2. Rocking Mode, Freight Element near Roof of Box Car - 50 mph
- Case 3. Bounce Mode, Freight Element at Front End of Box Car - 50 mph
- Case 4. Bounce Mode, Freight Element at Center of Gravity of the Car Body - 50 mph

8. CONCLUSIONS AND RESULTS | 43

APPENDIX

A. EQUATIONS OF MOTION | 45
B. COMPUTER PROGRAM LISTINGS | 75

C-1. EQUIVALENT VISCOUS DAMPING COEFFICIENT | 105
C-2. DETERMINATION OF SUSPENSION SPRING STIFFNESS FOR A TYPICAL TRUCK | 110
C-3. BENDING STIFFNESS OF A TYPICAL TRUCK BOLSTER | 110
C-4. DESCRIPTIVE DATA FOR A 70-TON BOX CAR | 113
C-5. TRACK INPUT EQUATIONS | 115

REFERENCES AND BIBLIOGRAPHY | 119
### LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1.</td>
<td>A Typical Freight Car Truck</td>
<td>7</td>
</tr>
<tr>
<td>3-1a.</td>
<td>The IIT Mathematical Model of a Freight Car with a Freight Element</td>
<td>12</td>
</tr>
<tr>
<td>3-1b.</td>
<td>Degrees of Freedom of the IIT Mathematical Model</td>
<td>13</td>
</tr>
<tr>
<td>6-1.</td>
<td>Car Body Roll Angle, 17.5 mph</td>
<td>29</td>
</tr>
<tr>
<td>6-2a.</td>
<td>Front Center Plate Vertical Reaction, 17.5 mph</td>
<td>29</td>
</tr>
<tr>
<td>6-2b.</td>
<td>Wheel Loads, 17.5 mph</td>
<td>29</td>
</tr>
<tr>
<td>6-3.</td>
<td>Front Bolster Lateral Reaction, 17.5 mph</td>
<td>30</td>
</tr>
<tr>
<td>6-4.</td>
<td>Front Center Plate Vertical Reactions - Bounce Mode, 60 mph</td>
<td>32</td>
</tr>
<tr>
<td>6-5.</td>
<td>Rear Center Plate Vertical Reactions - Bounce Mode, 60 mph</td>
<td>32</td>
</tr>
<tr>
<td>6-6.</td>
<td>Suspension Spring Group Compression, Right Front - Bounce Mode, 60 mph</td>
<td>33</td>
</tr>
<tr>
<td>6-7.</td>
<td>Suspension Spring Group Compression, Right Rear - Bounce Mode, 60 mph</td>
<td>33</td>
</tr>
<tr>
<td>7-1a.</td>
<td>Freight Element Lateral Acceleration at Roof of the Car (Rocking Mode - 17.5 mph)</td>
<td>36</td>
</tr>
<tr>
<td>7-1b.</td>
<td>Car Body Lateral Acceleration at its Center of Gravity (Rocking Mode - 17.5 mph)</td>
<td>36</td>
</tr>
<tr>
<td>7-2a.</td>
<td>Freight Element Acceleration Spectrum, Lateral (Rocking Mode - 17.5 mph)</td>
<td>38</td>
</tr>
<tr>
<td>7-2b.</td>
<td>PSD Analysis, Freight Element Lateral (Rocking Mode - 17.5 mph)</td>
<td>38</td>
</tr>
<tr>
<td>7-3a.</td>
<td>Lateral Acceleration of Freight Element at Roof of the Car (Rocking Mode - 50 mph)</td>
<td>39</td>
</tr>
<tr>
<td>7-3b.</td>
<td>Vertical Acceleration of Freight Element at Center of Gravity of the Car Body (Rocking Mode - 50 mph)</td>
<td>39</td>
</tr>
<tr>
<td>7-4a.</td>
<td>Vertical Acceleration of Freight Element at Front End of Car (Bounce Mode - 50 mph)</td>
<td>40</td>
</tr>
<tr>
<td>7-4b.</td>
<td>Vertical Acceleration of Freight Element at Center of Gravity of the Car Body (Bounce Mode - 50 mph)</td>
<td>40</td>
</tr>
</tbody>
</table>
Figure
7-5a. Freight Element Acceleration Spectrum, Vertical, Front End of the Car (Bounce Mode - 50 mph) ... 41
7-5b. PSD Analysis, Freight Element Vertical, Front End of the Car (Bounce Mode - 50 mph) ... 41
A-1a. Euler's Angles ................. 49
A-1b. Transformation of Body Angular Velocities to Inertial Angular Components ... 49
A-2. Bolster Lateral Constraint .......... 52
A-3. Vertical Springs at Car Body and Truck Bolster Interface ............... 54
A-4. Vertical Springs at Bolster and Truck Interface .... 57
A-5. Lateral Springs between Front Bolster and Truck ... 57
A-6. Track Springs at Front Truck ........... 59
C-1. Free Body Diagram of a Typical Friction Shoe during Upstroke ............... 108
C-2a. Schematic of the Cross-section of a Typical Truck Bolster ............... 112
C-2b. Area Moment of Inertia ............... 112
C-2c. Transverse Shear ............... 112
C-2d. Bending Moment Diagram ............... 112
C-3a. Rocking Mode Vertical Track Profile - Half-staggered ............... 117
C-3b. Bounce Mode Vertical Track Profile - Rail Joints in Phase ............... 117
C-3c. Lateral Track Profile ............... 117


**LIST OF TABLES**

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Degrees of Freedom</td>
<td>11</td>
</tr>
<tr>
<td>2.</td>
<td>Comparison of IIT, AAR and Stucki Models</td>
<td>26</td>
</tr>
<tr>
<td>3.</td>
<td>Comparison of Results with Stucki's - Rocking Mode</td>
<td>27</td>
</tr>
<tr>
<td>4.</td>
<td>Comparison of Results with Stucki's - Bounce Mode</td>
<td>34</td>
</tr>
</tbody>
</table>
## LIST OF ABBREVIATIONS AND SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>E&lt;sub&gt;1&lt;/sub&gt;</td>
<td>axle centers in each truck</td>
<td>inch</td>
</tr>
<tr>
<td>C</td>
<td>bolster center of gravity above that of the truck</td>
<td>inch</td>
</tr>
<tr>
<td>CF</td>
<td>freight cushion material damping coefficient</td>
<td>lb-sec/in</td>
</tr>
<tr>
<td>C&lt;sub&gt;i&lt;/sub&gt;</td>
<td>damping coefficient at point i</td>
<td>lb-sec/in</td>
</tr>
<tr>
<td>C&lt;sub&gt;i&lt;/sub&gt;L</td>
<td>lateral damping coefficient at point i</td>
<td>lb-sec/in</td>
</tr>
<tr>
<td>D</td>
<td>truck center distance</td>
<td>inch</td>
</tr>
<tr>
<td>d&lt;sub&gt;1&lt;/sub&gt;</td>
<td>horizontal distance of center plate from car body center of gravity</td>
<td>inch</td>
</tr>
<tr>
<td>d&lt;sub&gt;2&lt;/sub&gt;, d&lt;sub&gt;3&lt;/sub&gt;</td>
<td>longitudinal distance between center of gravity of the bolsters from suspension spring groups</td>
<td>inch</td>
</tr>
<tr>
<td>d&lt;sub&gt;4&lt;/sub&gt;, d&lt;sub&gt;5&lt;/sub&gt;</td>
<td>longitudinal distance between center of gravity of the trucks from suspension spring groups</td>
<td>inch</td>
</tr>
<tr>
<td>e</td>
<td>center plate radius</td>
<td>inch</td>
</tr>
<tr>
<td>g</td>
<td>acceleration due to gravity</td>
<td>in/sec²</td>
</tr>
<tr>
<td>g&lt;sub&gt;1&lt;/sub&gt;</td>
<td>distance between centerline of bolster and sidebearings</td>
<td>inch</td>
</tr>
<tr>
<td>GAP</td>
<td>sidebearing clearance</td>
<td>inch</td>
</tr>
<tr>
<td>GIB</td>
<td>gib clearance</td>
<td>inch</td>
</tr>
<tr>
<td>H</td>
<td>half rail gauge</td>
<td>inch</td>
</tr>
<tr>
<td>h&lt;sub&gt;2&lt;/sub&gt;, h&lt;sub&gt;3&lt;/sub&gt;</td>
<td>lateral distance between bolster center of gravity and suspension groups</td>
<td>inch</td>
</tr>
<tr>
<td>I&lt;sub&gt;xi&lt;/sub&gt;</td>
<td>rotational inertia of mass i about x axis</td>
<td>lb-in-sec²</td>
</tr>
<tr>
<td>I&lt;sub&gt;yi&lt;/sub&gt;</td>
<td>rotational inertia of mass i about y axis</td>
<td>lb-in-sec²</td>
</tr>
<tr>
<td>I&lt;sub&gt;zi&lt;/sub&gt;</td>
<td>rotational inertia of mass i about z axis</td>
<td>lb-in-sec²</td>
</tr>
<tr>
<td>KBOM</td>
<td>bottomed spring stiffness</td>
<td>lb/in</td>
</tr>
<tr>
<td>KF</td>
<td>freight spring stiffness</td>
<td>lb/in</td>
</tr>
<tr>
<td>K&lt;sub&gt;i&lt;/sub&gt;</td>
<td>vertical spring stiffness at point i</td>
<td>lb/in</td>
</tr>
<tr>
<td>K&lt;sub&gt;i&lt;/sub&gt;L</td>
<td>lateral spring stiffness at point i</td>
<td>lb/in</td>
</tr>
<tr>
<td>KP&lt;sub&gt;ij&lt;/sub&gt;</td>
<td>pitching spring stiffness between bodies i and j</td>
<td>lb/rad</td>
</tr>
<tr>
<td>KT&lt;sub&gt;ij&lt;/sub&gt;</td>
<td>torsional spring stiffness between bodies i and j</td>
<td>lb/rad</td>
</tr>
<tr>
<td>m&lt;sub&gt;i&lt;/sub&gt;</td>
<td>mass of body i</td>
<td>lb-sec²/in</td>
</tr>
<tr>
<td>R</td>
<td>center of gravity of wheel sets above rails</td>
<td>inch</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Units</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------</td>
<td>-------</td>
</tr>
<tr>
<td>Rg</td>
<td>rail lateral inputs</td>
<td>inch</td>
</tr>
<tr>
<td>Rl</td>
<td>rail lateral inputs</td>
<td></td>
</tr>
<tr>
<td>RL</td>
<td>rail lateral inputs</td>
<td>inch</td>
</tr>
<tr>
<td>S</td>
<td>maximum rail surface variation</td>
<td>inch</td>
</tr>
<tr>
<td>TL</td>
<td>spring travel to solid</td>
<td>inch</td>
</tr>
<tr>
<td>Vl to</td>
<td>rail vertical inputs</td>
<td>inch</td>
</tr>
<tr>
<td>Vs</td>
<td>rail vertical inputs</td>
<td>inch</td>
</tr>
<tr>
<td>xi</td>
<td>lateral displacement of mass i</td>
<td>inch</td>
</tr>
<tr>
<td>yi</td>
<td>longitudinal displacement of mass i</td>
<td>inch</td>
</tr>
<tr>
<td>zi</td>
<td>vertical displacement of mass i</td>
<td>inch</td>
</tr>
<tr>
<td>psi</td>
<td>roll angle of mass i</td>
<td>radian</td>
</tr>
<tr>
<td>phi</td>
<td>pitch angle of mass i</td>
<td>radian</td>
</tr>
<tr>
<td>alpha</td>
<td>yaw angle of mass i</td>
<td>radian</td>
</tr>
</tbody>
</table>
1. INTRODUCTION TO FREIGHT DAMAGE ASPECTS

Freight loss and damage faced by the rail transportation has been a very serious problem in the United States. For the year 1973, a total of $232,576,501 was claimed for L & D (1-1). This amount, being 0.8% less than that reported in 1972, was still very sizable compared with the earnings of the rail transportation of freight.

Increasingly, attempts have been made to identify the causes and to solve the principal freight damage problems. In the recent years, extra emphasis has been placed on the aspect of careful freight handling, by the manufacturers and the railroad shippers. Proper loading methods for different freight in a railroad car are recommended by the Freight Loading and Container Section of the Association of American Railroads (1-2). Defective or unfit equipment in railroad cars will render the freight more susceptible to damages. Shippers are urged to replace such equipment when necessary. Moreover, the dynamics of the car and that of the freight/package system contribute significantly to much of the direct or indirect causes of freight damage.

The dynamics of the car are primarily of two kinds—longitudinal shocks and over-the-road type of vibrations. Longitudinal shocks on the cars due to coupling and humping actions in classification yards give rise to high acceleration level impulse type of excitations to the freight within the car. Most of today’s package design criteria are aimed at protecting freight elements from such shocks and usually drop tests are used to study and subsequently to choose certain cushioning materials as for packaging a particular kind of freight. Over-the-road vibrations, although usually do not give rise to high amplitudes of excitation, may have a prolonged effect on the freight elements. Moreover, package materials designed to protect the freight primarily from shocks may turn out to be providing bad dynamic environments to the products within, (1-3) while the train is running over the tracks. If a resonant frequency of the freight and cushioning material combination is encountered, the cushioning material may in effect magnify the input vibration levels to
the freight and cause damage instead of protecting it. Vibrations, on the other hand, may not necessarily be the direct cause of freight damages. Prolonged vibrations can shake loose the freight elements from their initial loading positions. This loosening effect, which is called rattle, allows free motion of the freight element. This, coupled with the longitudinal shocks during run-ins and run-outs (depending on the type of terrain that the train is running through) can cause tumbling and falling of freight elements and subsequently damage them. It can be seen then that a severe vibration environment may contribute very much to freight damage both directly and indirectly. The indirect contribution is made when vibrations establish a damage-prone situation. Thus it becomes clear that for prevention of freight damage it is necessary to fully understand the vibration environment in a typical rail car and then isolate the freight from such vibrations as much as economically possible.

In the past years, tremendous emphasis has been given to the handling aspects by shippers and the rail industry. It is evidently very important now to study the dynamics of the car together with that of the freight/package systems. This study can enable better future design of packaging materials for the dynamic requirements of freight in a railroad car. Furthermore, a study as such may yield better design criteria for the railroad truck suspension, damping systems and track requirements for a smoother ride for the freight. At this stage of research, it is our primary interest to study the freight environment in a typical railroad car.

One effective way to study the freight environment in a rail car which allows the option of modifying and redesigning fairly easily the various parameters such as suspension and damping systems, etc., is by the mathematical model study approach. Several mathematical models have been developed in this field and used in simulation studies of various aspects of the freight car dynamics (1-4, 1-5, 1-6, 1-7, 1-8, 1-9, 1-10, 1-11, 1-12). Each has made contributions toward solutions of dynamic problems in the field. Industry models and detailed
information on them is not freely available because of proprietary rights. Other models, while good for individual tasks for which they were developed were not deemed suitable for determination of the three dimensional coupled car-freight element dynamics problem being addressed here.

A comprehensive study of freight dynamics necessitates a simulation of truck, car and freight components, all in one mathematical model, so that the interactions between these various components can be properly represented. The various mathematical models set up by the Stucki Company (1-10), Patel/Martin (1-11) and Tse/Martin (1-12) address the dynamics of rail car as a complete unit, but each is directed towards a specific goal. For example, the Stucki Company analysis attempts to establish the damping requirements to control vertical and roll motions of rail cars, and does not incorporate effects of these motions on freight elements. The model developed by Patel/Martin is relatively simple and explores a basic method of approach to computer simulations of rail cars. The Tse/Martin model was only recently developed, and was an independent parallel study conducted at the Association of American Railroads, under the Track Train Dynamics Program. It describes the rail car and truck using a 20-degree of freedom model and simulates the car body as a flexible (2 mass) system. The model was specifically developed for use in parametric studies on the dynamic characteristics of rail cars. The IIT simulation developed in the present study includes the coupling of freight elements, car body, truck motions and track characteristics in one mathematical model.

The mathematical-computer model developed here at IIT is the first known solution incorporating the dynamics of the freight car truck, the bolsters, the car body and the freight element into one comprehensive analysis. It has been developed from a very basic and simple model, through several iterations of the design cycle, up to the current configuration, which has been shown from a dynamics point of view, to be a good representation of a real freight car system.

The report given here is to describe the work conducted in
developing this mathematical model, and to describe the details of the model.
2. SELECTION OF A CAR AND DETERMINATION OF ITS CHARACTERISTICS

The dynamics of the car and that of the freight/package system may be studied by mathematically modeling the system and performing simulations on the model. For maximum value of such a study to the American Railroad Industry, the model should be based on a car and its truck that is widely used in North America. The majority of commodities shipped by rail today are carried inside closed cars (2-1). The dynamic environment in these cars then can be considered as representative of the service environment encountered by freight packages. In a recent "over the road" test program (for determining the acceleration levels and forces in various parts of a box car under service conditions) performed by A.A.R. (2-2), a 70-ton box car with a typical truck was selected as a representative vehicle, to run the test over 5000 miles on different railroads. Based on this information, a 70-ton box car with a typical truck is chosen here for the basis of the mathematical model. It is highly representative of the North American railcars. In addition, future correlation studies with the 5000 Mile Road Test data may be conducted, and the 'ground work' for this has now been completed.

After selection of a car and truck, work was conducted to identify the significant components of the car/truck system and to determine their characteristics. As originally proposed, the approach adopted here has been to model the car as simple as possible for the initial analysis, and only when this model is properly functioning, and its simulation possibilities fully explored, then the additional complexities involved in improving simulation have been added in a stepwise manner. Initially then, the car had to be regarded as a five mass system, with linear springs and dampers. Nonlinear effects (spring bottoming, clearances, etc.) have subsequently been added. At the present time, the development of the mathematical model components is as follows.

2.1 The Car Body

As shown by the Freight Car System tests of Nasa-Martin Marietta Corp., a rigid car is a good representation of a current
typical U.S. box car, that is, the flexibility of the box car body may be considered negligible. In such a case, the mathematical representation may be made as a rigid body, with its mass concentrated at the center of gravity. It is, therefore, appropriate in the present model to assume a single lumped mass for the car body. This requires that the following parameters be determined: mass and mass moments of inertia about the three principal axes, location of the center of gravity of the car body, width of the car and other geometric parameters determined from referenced literature and/or railroad publications (for details refer to Appendix C).

2.2 The Center Plates

These are locations for support of the car body weight on the truck bolsters. They are the circular plates of diameters 13-14 inches about which the car body swivels. Lubrication pads are introduced between the center plates to reduce friction. Relative vertical motions between the car body and truck bolsters may cause center plate separation. Severe car roll motion may result in only partial contacts of the center plates. For all practical purposes, then, center plates may be modeled as two stiff vertical springs.

2.3 The Side Bearings

These are the stoppers located on the upper surface of the truck bolsters. Some of them may be spring loaded and some are just rollers. Their function is to inhibit the car from rolling about its longitudinal axis indefinitely. If the weight of the car body plus that of the freight is fairly evenly distributed, all side bearing clearances statically are of the order of 1/4 inch. When the car rolls severely, this clearance may vanish and the car load is shared by the center plate and the side bearings. This nonlinear effect is modeled into the computer simulation.

2.4 The Center Plate Extension Pads (C-PEP Pads)

On more recent designs of bolsters by some railcar manufacturers, a new feature called the C-PEP Pads is incorporated at a location between the center plate and the side bearings.
These pads are made of a certain elastomeric material that provides horizontal rotational control in addition to complementing the vertical damping.

In order not to further complicate the mathematical model at this time, this effect is reserved as an option for further modifications on the model. At present, bolsters without this feature constitute the more conventional designs which are still widely used in the rail industry of today.

2.5 The Axle, Wheel Set and Sideframes

These are all rigid masses and have been assumed to be equivalent to one lumped mass, located at the center of gravity of the truck. Geometric dimensions of each have been determined.

2.6 Suspension Springs

The typical truck with 3 11/16 inch travel linear springs are studied from mechanical drawings and their stiffnesses determined. (For details see Appendix C.)

2.7 The Friction Shoe and Spring-Steel Wear Plate

Together these form a typical damping system, and the friction shoe can be spring loaded to give variable force depending on the loading and the direction of the damping stroke. From some railroad company experimental studies, it was found that the frictional force developed on the wear plate for the upward stroke was different from that for the downward stroke. The energy dissipated by the upward stroke is 65% of the complete cycle. This information was made use of in later deriving the equivalent viscous damping coefficient (see Appendix C).

It became apparent that the modeling of truck damping systems is possible as both Coulomb and/or viscous, and that perhaps an equivalent viscous damping approach would be fruitful, allowing this model to be compared with both existing (Coulomb) damping systems, and at the same time, used for future design and evaluation of hydraulic dampers. Consequently, a new analytical method for incorporating freight car truck damping effects into the mathematical model has been developed, based on an equivalent viscous damping concept. The computer model now has the capability of either Coulomb, viscous or equivalent viscous dampings.
2.8 Rail Profile and Subgrade Structures
   Noting that these are not rigid structures, the elasticity of the subgrade has to be considered (2-3), and is modeled into this computer program as a vertical and lateral road bed stiffness.

2.9 Couplers
   The car is subjected to longitudinal coupler inputs depending on the type of truck, track condition, terrain, coupler design, etc.
   
   Note: Details of numerical values on spring stiffness, equivalent viscous damping coefficient, and descriptive data for a 70-ton box car are reported in Appendix C.
3. THE MATHEMATICAL MODEL

3.1 General Description

In the mathematical modeling of the freight car, a 27-degree of freedom nonlinear model has been developed. This simulates a 70-ton box car as 6 lumped masses connected by spring damper groups, and each mass has several degrees of freedom. The 6 masses respectively represent the car body, the front and rear bolsters, the front and rear side frame/axle sets (trucks), and an element of freight.

The degrees of freedom modeled for the various masses are shown in the table below.

<table>
<thead>
<tr>
<th>Freight car element</th>
<th>Degree of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Translational</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Car body</td>
<td>X</td>
</tr>
<tr>
<td>Front bolster</td>
<td>X</td>
</tr>
<tr>
<td>Rear bolster</td>
<td>X</td>
</tr>
<tr>
<td>Front truck</td>
<td>X</td>
</tr>
<tr>
<td>Rear truck</td>
<td>X</td>
</tr>
<tr>
<td>Freight element</td>
<td>X</td>
</tr>
</tbody>
</table>

3.2 Car Body and Truck Bolsters Interface

3.2.1 Center Plates. These are modeled here as four vertical springs, each with a very high stiffness. This allows for relative vertical motions between the car and the truck bolsters. At this time, relative lateral motions between the car body and the two truck bolsters are assumed constrained to be zero.

3.2.2 Sidebearings. The sidebearing reaction is modeled as a spring with a clearance between the car body underframe and the spring. It does not have any stiffness until there is no clearance.

3.3 Truck Bolsters and Sideframes Interface

3.3.1 Suspension Springs. A total of four vertical springs is modeled per truck, with two springs on each side. In order to
FIG. 3-1A. THE IIT MATHEMATICAL MODEL OF A FREIGHT CAR WITH A FREIGHT ELEMENT
FIG. 3-1b. DEGREES OF FREEDOM OF THE IIT MATHEMATICAL MODEL
incorporate the bending effect of the bolsters, an analysis was made to estimate the bending stiffness of the bolster (see Appendix C). Consequently, the values of the stiffness used for suspension springs are the effective spring stiffness of the suspension group in series with the bending stiffness of the truck bolster. Two additional nonlinear springs are used to model the effect of spring bottoming, which happens at severe bounce situations.

Although the actual suspension springs are in the vertical plane, when the truck bolster moves laterally relative to the sideframe, certain lateral elastic constraint is introduced into the system. These spring actions are accounted for in this mathematical model by four lateral springs per truck.

3.3.2 Friction Plates. These are the energy dissipation elements used in the truck for damping down the amplitudes of the vibratory motions of the truck. In this mathematical model the friction plates are represented as either an equivalent viscous damping (see Appendix C) or Coulomb's friction damping. Incorporating viscous type damping models into the computer simulation at this time is considered advantageous since it both simplifies the mathematics and in addition leaves the way open for future investigations of modified damper designs. The Computer Program developed here (see section on Computer Program) allows the option of running the computer simulation with either dry or the equivalent viscous damping. Consequently, it has not lost the reality of dry friction type damping, used on the majority of today's trucks.

3.3.3 Gib Clearance. This is the lateral clearance between the bolster and the sideframe. A nonlinear spring with a very high stiffness is used for modeling the gib effect.

3.4 Rail Profile and Subgrade

Depending on the quality of the subgrade under the rails, and the dynamics of the railcar, up to several inches of rail depressions have been observed when a train passes over it. This effect is modeled as eight track springs in each of the vertical and lateral directions, located at the contact points
between the wheels and the rails. The North American Continent rail tracks are usually half-staggered (i.e., the rail joints on one side of the track is at mid-length of the rail on the opposite track) and depressions at the rail joints can conservatively be estimated to be of the order of 3/4 inch. Previous studies, made by other railroad researchers (2-3) showed that the profiles of some of the revenue tracks actually look like "rectified sine waves". From this, the rail profile used in this simulation is that of a rectified sine wave, with an amplitude of 3/4 inch vertically, and 3/8 inch laterally, in phase (Figure C-3c).

3.5 Coupler Forces

When trains are moving over hilly areas, run-ins and run-outs can create a very significant shock and vibratory environment inside the car. Longitudinal coupler inputs can be modeled as if they act at the car body center of gravity by introducing an equivalent force and couple combination at the center of gravity. The spring-damper model used here for this purpose is excited by the equivalent force only, in order to verify that the model functions properly. Rotational responses due to couples at the center of gravity are checked out when the model is excited in the roll or the bounce modes, and so does not need to be input for this purpose at this point. Other severe coupler shock forces are also developed in service, due to the freight yard humping action. This depends, among other things, on the speeds at which the cars are coupled together (3-1). At this stage of the research, only the vibratory coupler inputs are being considered.

3.6 Car and Freight Element Interface

A linear spring-damper system is again assumed. The dynamic damping characteristics have been obtained experimentally as part of this total project here at Illinois Institute of Technology. This part of the program will be reported in a separate Project Technical Report.

It can be seen from the discussion given above, that the mathematical model developed here is nonlinear, as in the actual rail car. Gib clearances, sidebearing clearances and the spring bottoming effects are some of the nonlinearities in the system.
These are simulated by the system of equations developed, which are then solved on the digital computer, yielding displacements, velocities and accelerations at any point(s) in the freight car. Dynamic loadings at the center plates, truck bolster lateral reactions and wheel loads, wheel lifts, suspension spring deflections plus all the forces and reactions in the system can be studied in the present model.
4. EQUATIONS OF MOTION AND METHOD OF SOLUTION

4.1 Equations of Motion

After setting up the mathematical model, equations of motion have to be derived and solved for accelerations, velocities and displacements (both rotational and translational).

For a simple spring-mass-damper system in which the body is under excitation from an external force \( p \), the equation of motion is

\[
m\ddot{x} + c\dot{x} + kx = p,
\]

where
- \( m \) - mass of the body
- \( c \) - viscous damping coefficient
- \( k \) - spring stiffness.

If \( m, c, k, p, \dot{x} \) and \( x \) initial are prescribed, the acceleration \( \ddot{x} \) can be computed.

For more complicated spring-mass-damper systems such as the one representing the 70-ton box car with the typical truck, a systematic way of writing the equations of motion step by step without much complication is the Lagrange's Equation of motion which can have the form

\[
\frac{d}{dt} \left( \frac{\partial E}{\partial \dot{q}_i} \right) - \frac{\partial E}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} = Q_i,
\]

where
- \( E \) - kinetic energy of the system
- \( V \) - potential energy of the system
- \( D \) - Rayleigh's dissipation function of the system
- \( Q_i \) - generalized forces, and
- \( q_i \) - generalized coordinates \( i = 1, 2, \ldots, 27 \) \((x_1, y_1, z_1, \psi_1, \phi_1, \alpha_1, z_2, \psi_2, \phi_2, \alpha_2, z_3, \psi_3, \phi_3, \alpha_3, x_4, z_4, \psi_4, \phi_4, \alpha_4, x_5, z_5, \psi_5, \phi_5, \alpha_5, x_6, y_6, z_6)\)

With 27 generalized coordinates, there are 27 equations of motion similar to the one in eq. (4-1). Details of the derivation of these equations are listed in Appendix A Equations of Motion.
Some of the equations derived are coupled with one another and they are grouped into 5 different matrices. All the equations, both coupled and uncoupled are programmed (in Fortran language) and then solved for 27 accelerations simultaneously. Integrating these accelerations twice leads to the corresponding velocities and displacements.

4.2 Solution by Computer Iterative Method

For dynamic simulation on the computer model, an iterative method has been developed, which uses the UNIVAC 1108 computer (on campus). Rail surface variations as well as coupler inputs are sources of excitation. Simulation is started at time equals zero, with the system at static equilibrium. Excitation is applied to the model and the resulting accelerations computed. The coupled accelerations are solved by a subroutine LSIMEO (which is currently available in the Math Pak at the Information Processing Center of Illinois Institute of Technology, and the listing of which is given in Appendix B Computer Program Listings). All accelerations are then numerically integrated by the Runge-Kutta Integration technique to obtain velocities and displacements at the center of gravity of the masses in the model. These new velocities and displacements, together with the excitation, are the values based upon which the accelerations of the second time step are computed. This process is repeated for each time step and new values are computed on old ones. An iteration method of this nature enables us to study the dynamic responses of the system up to any desired period of simulation.
5. COMPUTER PROGRAM

A computer program has been developed for solving the equations and simulating the dynamic responses of the box car/freight system. This program consists of a Main program and nine subroutines. The function and philosophy of each is discussed below.

5.1 MAIN Program

This program functions as the coordinator for the subroutines. It calls the subroutines ACCELN and RUNG to obtain values of acceleration, velocities and displacements at centers of gravity of the masses in the model. Geometric parameters such as truck center distance, length of the various components of the car and truck, spring stiffnesses, and moments of inertia, etc., are entered in this portion of the program.

5.2 Subroutine DELGAP

At each time step, this subroutine computes all four of the sidebearing clearances between the body bolsters and the truck bolsters. If a sidebearing is touching the body bolster, there is an additional reaction at the point of contact other than those at the center plate locations. This reaction is accommodated by adding, at this time step, an additional spring in parallel with the center plate vertical springs. This additional spring will remain in effect for as long as there is no sidebearing clearance.

5.3 Subroutine DELGIB

Between the truck bolster and the column of the sideframe, there is a small clearance (of the order of 3/4 inch) laterally. This is the gib clearance. When the box car is running on the rails, lateral motions exist such that the truck bolster may be hitting the column. This zero gib clearance effect is modeled here by adding a gib spring in parallel with the lateral springs whenever such condition exists. This is done by first checking the gib clearance. A coefficient denoted by the symbol δ is utilized to facilitate the modeling of the spring nonlinearity. A value of 1 or 0 is correspondingly assigned to δ to bring in the gib spring action or remove it. A total of 4 gib springs is used to model the gib effect on the two truck bolsters.
5.4 Subroutine SPRING

During severe conditions of the bouncing mode, especially with heavily loaded cars, spring bottoming phenomenon is not uncommon. In order that the computer model can simulate and predict the suspension spring bottoming effect, a SPRING subroutine is developed for this purpose. At every time step, the vertical motions of the truck bolster relative to those of the sideframe are checked against the allowable travel length of the spring as specified by designers and manufacturers. If it is found that any suspension spring group reaches its solid length, a comparatively high stiffness spring is introduced in the vertical model to stop the truck bolster from compressing on the suspension springs very much further. This high stiffness spring will be removed by the subroutine if spring bottoming does not exist.

5.5 Subroutine ACCELN

The function of this subroutine is to compute the acceleration values at each iteration. First, it calls the subroutines DELGAP, DELGIB and SPRING to update the total number of springs suitable for use at each iteration time step. For example, if after calling the three subroutines it is found that two of the four sidebearing clearances reduce to zero and two suspension groups bottom out but the truck bolster are not hitting the column of the sideframe, then two additional sidebearing springs and two bottoming springs but no gib springs will be included in the current iteration to calculate the new acceleration values.

Sixteen of the acceleration variables are coupled (e.g. those of the rotational coordinates) and they are grouped into five matrices. This grouping into smaller matrices, rather than retaining all the coupled equations in one big matrix, significantly helps to speed up the solving process by LSIMEO. The rest of the acceleration variables are computed independently.

ACCELN also calls other subroutines CPLATE, CAL and T5000, the functions of which will be discussed in their own sections later in this report.

5.6 Subroutine RUNG

After accelerations are computed by ACCELN, their values are
transferred back to the main program. Then MAIN calls subroutine RUNG to integrate numerically the accelerations twice yielding the corresponding velocities and displacements.

Subroutine RUNG is a standard program available in most computer-aided numerical analysis books. The one used here is a fourth order Runge-Kutta method which predicts the value of $y_{i+1}$ from the value of $y_i$ at time $t_i$ based on the formula:

$$y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where $K_1$, $K_2$, $K_3$, and $K_4$ are weighted averages as follows:

$$K_1 = f(y_i, t_i)$$
$$K_2 = f(y_i + \frac{h}{2}K_1, t_i + \frac{h}{2})$$
$$K_3 = f(y_i + \frac{h}{2}K_2, t_i + \frac{h}{2})$$
$$K_4 = f(y_i + hK_3, t_i + h)$$

and $h$ is the step size. Note: The fourth order Runge-Kutta technique gives as accurate value of $y_{i+1}$ as a fourth order Taylor series. The choice of the step size $h$ has to depend on the response frequency of the system studied. The following exercise serves as an example on estimating the step size.

Estimation on the time step for using Runge-Kutta technique based on the natural frequency of the bolster:

Total lateral spring stiffness = 4(4425 + 666000) lb/in. = 2,681,700 lb/in.

Mass of the bolster = \frac{1150}{g}

Natural frequency $f_n = \sqrt{\frac{K}{M}}$

$$= \sqrt{\frac{2,681,700 \times 386.4}{1150}} \text{ Hz}$$

$$= 950 \text{ Hz}$$

Periodic time = $\frac{1}{f_n} = \frac{1}{950} \text{ sec.} = .00105 \text{ sec.}$

Rule of thumb: 8 iterations is minimum per response cycle for using Runge-Kutta
Time step = $\frac{0.00105}{8}$ sec.

= 0.00013 sec.

This analysis can be considered as a rough guide to estimate the order of the step size. However, the optimum value still needs to be obtained through experience and trial and error.

5.7 Subroutine CPLATE

This subroutine computes the vertical reaction of the front and rear center plates.

5.8 Subroutine CAL

Each time this subroutine is called it computes the front bolster lateral reaction as well as all the wheel loads. This subroutine gives valuable information as to the loading on the column of the sideframe and the phenomenon of wheel-lifts, the severe cases of which may cause derailments.

5.9 Subroutine T5000

This subroutine computes various accelerations at selected locations within the dynamic system, based on the accelerations of the rigid body masses and the geometry of the rigid bodies. In the analysis performed so far, the vertical and lateral accelerations on the roof and the floor of the box car at the front and rear ends of the car have been computed. These specific locations have been chosen to yield data for later comparison with field data obtained by the 5000 MILE BOX CAR VIBRATION TEST.

5.10 Subroutine SGNFUN

A special feature introduced in this computer program is the option of simulating the suspension damping system either as an equivalent viscous damping or as Coulomb's damping. If the latter is preferred, one statement is added in the subroutine ACCLN to call on SGNFUN.

Subroutine SGNFUN computes, at every time step, the relative velocities, both vertical and lateral, between the bolsters and sideframe columns. It then assigns a positive or a negative sign to the constant damping force such that the damping force always opposes the motion.

With such an option, two simulations can be run on the
computer using the Coulomb's friction or equivalent viscous damping without any major modifications on the program at all. This is of great value in both development of new suspension damping systems or in evaluation studies of existing systems. Hence, this method of modeling the damping system of the typical truck as an equivalent viscous damping can readily be justified.

5.11 Nonlinear Modeling

It can be seen that in this computer simulation a great number of nonlinear springs can now be introduced by DELGAP, DELGIB and SPRING subroutines. These effects have been added to the model in a stepwise manner, and this has added a considerable amount of complexity to the computer program. It is felt, however, that the nonlinear model so developed is now a close simulation of a railcar dynamic system, and the complexity is justified. The correlation of this model with other data which has been verified by tests is discussed later in this report, and this nonlinear modeling of the railcar is shown to be validated.

5.12 Computer Program for Frequency Analysis

The main program and the nine subroutines discussed so far give as output data on acceleration, velocity and displacement at any point(s) in the car and the truck, in addition to the dynamic loadings on the car structure. These output data are given as a function of time. However, road test data often express acceleration levels (g) in the frequency domain. The frequency analysis on excitation levels becomes more meaningful to designers of freight cushioning and packaging materials, and also useful to the rail car and truck designers. With this in mind, the program FREQ was developed.

From our computer simulation results based on the rectified sine wave vertical rail inputs as well as the Stucki Company's data, we observed that the accelerations of the car body and those of the freight are quite periodic. In order to study the frequency contributions of these acceleration responses, one convenient method is to apply the Fourier analysis. Any periodic motion (including those which are complex) can be represented by
a series of sines and cosines which are harmonically related. For example, if \( x(t) \) is a periodic function of the period \( T \), it can be represented by the Fourier series

\[
x(t) = \frac{a_0}{2} + a_1 \cos \omega_1 t + a_2 \cos 2\omega_1 t + \ldots
\]

\[
+ b_1 \sin \omega_1 t + b_2 \sin 2\omega_1 t + \ldots
\]

where \( \omega_1 = \frac{2\pi}{T} \) is the fundamental frequency and the coefficients \( a_0, a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) completely define the harmonic contribution of the period wave. With some manipulation of algebra, the coefficients can be obtained by hand calculation. In order to minimize the computational time, the digital computer is again used.

A program has been developed by the Information Processing Center at Illinois Institute of Technology to specifically compute the coefficients for the Fourier series. This we incorporate as a subroutine in our Frequency Analysis program, FREQ, and have it compute the amplitudes of contribution at each of the harmonics. Then the Fourier spectrum of the wave form can be plotted. Since some of the road test data from the 5000 Mile Box Car Vibration Test will eventually be presented as PSD (Power Spectral Density) functions, the computer program developed here in this project can output data suitable for PSD analysis. This will be useful in later correlation studies.

At present, FREQ is an independent computer program from the MAIN and its nine subroutines. Future work is planned to incorporate FREQ as an additional subroutine to MAIN so that we can run rail car freight dynamic simulations on our model with output acceleration levels in the time and/or frequency domain(s).

Note: All program listings are in Appendix B Computer Program Listings.
6. COMPUTER MODEL VALIDATION

The computer programs have been thoroughly debugged to give the correct logic and mathematical solutions. However, in order to show that the computer model properly simulates a real rail car dynamic system, it is necessary at this stage to validate the computer model. This ensures that any future dynamic analysis based on this model for design purposes will be meaningful.

6.1 Methods of Validation

Validation can be done in two ways. First, other computer model outputs can be checked against this model. One such computer model in the rail industry now is the model that Stucki Company developed (1-10) to study the damping requirements to control vertical and roll motions of freight cars. This A. Stucki model has been validated against test data. The model is proprietary to the Stucki Company and is not freely available. However, some output data based on this model has been published (1-10) and can be of value to other workers for validation purposes. Recently, the Association of American Railroads, as part of the Track Train Dynamics Programs, have developed a computer model (1-11) in which the box car body is modeled as two lumped masses joined by torsional springs to incorporate flexibility. These models, among numerous others, can be used to help establish validity for the model developed here, even though many significant differences in these various models exist.

As a second method of validation, the computer model simulation outputs can be correlated with field test data generated by other researchers. The most recent and extensive piece of road test data on shock and vibration service conditions was completed by the Research and Test Department of the Association of American Railroads (2-2). In this test, a 70-ton box car with a typical truck was placed in actual revenue trains on a number of railroads. Data so collected will reflect a wide range in speed, track characteristics and operating terrain. Since the type of car and track for this test is the same as modeled here, it will be very beneficial to try to correlate with such test data when it is available. Currently, data from this test pro-
gram is being prepared by AAR for spectral and RMS analysis, and it is intended to further validate the IIT mathematical model with this data when it has been reduced to the appropriate form.  

6.2 Comparison of IIT Model with Other Models

Before comparing the IIT computer simulation outputs with others, it is desirable to consider the following table, which briefly summarizes the similarities and differences between this model and some others.

Table 2. Comparison of IIT, AAR and Stucki Models

<table>
<thead>
<tr>
<th></th>
<th>AAR</th>
<th>STUCKI</th>
<th>IIT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degrees of freedom</td>
<td>20</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>Rigid car body</td>
<td>2 masses</td>
<td>1 mass</td>
<td>1 mass</td>
</tr>
<tr>
<td>Damping model</td>
<td>Friction</td>
<td>Friction and Viscous</td>
<td>Friction or Equivalent Viscous</td>
</tr>
<tr>
<td>Freight Element</td>
<td>-</td>
<td>-</td>
<td>Lat, Vert, Long</td>
</tr>
<tr>
<td>Model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Track input type</td>
<td>Vert, Lat</td>
<td>Vert</td>
<td>Vert, Lat</td>
</tr>
<tr>
<td>Options:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Center plate</td>
<td>-</td>
<td>-</td>
<td>Yes</td>
</tr>
<tr>
<td>extension pads</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Longitudinal input</td>
<td>-</td>
<td>-</td>
<td>At car body</td>
</tr>
</tbody>
</table>

For all simulations on the IIT computer model, a rectified sine wave has been used as vertical input displacements on the wheels with maximum rail surface variation equal to 3/4 inch. At this time, the lateral rail input is assumed to exist on one track only with maximum variation equal to half of that of the vertical. Details of track input equations are found in Appendix C.

6.3 Validation of IIT Model

To study the dynamic responses of the box car/freight system for the rocking mode, the conventionally half-staggered rail joints are adopted in the simulation. Different car speeds were simulated to identify the critical speed, which is 17.5 mph for our system, compared with 15 mph for the Stucki model system.
The following output data were plotted and compared with A. Stucki Company data on a 100-ton hopper car for purposes of comparison of the wave shapes of the dynamic responses for the two systems modeled.

(a) Car body roll angle (Fig. 6-1)
(b) Front center plate vertical reaction (Fig. 6-2a)
(c) Wheel loads (Fig. 6-2b)
(d) Front bolster lateral reaction (Fig. 6-3)

The following table lists the comparisons of results of computer simulations for the rocking mode.

<table>
<thead>
<tr>
<th></th>
<th>STUCKI (100-ton hopper car)</th>
<th>IIT (70-ton box car)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Max. roll angle P-P</td>
<td>11.5° (15 mph)</td>
<td>11.4° (17.5 mph)</td>
</tr>
<tr>
<td>2. Time occurred (from</td>
<td>9.4 sec</td>
<td>8.6 sec</td>
</tr>
<tr>
<td>beginning of simulation)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Max. front center plate</td>
<td>160 K lb</td>
<td>105 K lb</td>
</tr>
<tr>
<td>loading</td>
<td>5.2 sec</td>
<td>5.0 sec</td>
</tr>
<tr>
<td>4. First occupancy of center</td>
<td>40 K</td>
<td>45 K*</td>
</tr>
<tr>
<td>plate separation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Max. front bolster</td>
<td>145 K lb</td>
<td>138 K lb</td>
</tr>
<tr>
<td>lateral reaction</td>
<td>6.0 sec</td>
<td>5.4 sec</td>
</tr>
<tr>
<td>6. Max. wheel loads -2</td>
<td>0.3-0.5 sec</td>
<td>0.2-0.4 sec</td>
</tr>
<tr>
<td>wheels (left front)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. First occurrence of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>wheel lift (left front)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Duration of wheel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>lift (left front)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Lateral bolster loadings for IIT model result from both vertical and lateral rail excitations. This value is slightly higher than that of the Stucki's which is based on vertical input only.
FIG. 6-1. CAR BODY ROLL ANGLE (DEGREES)
FIG. 6-2A. FRONT CENTER PLATE VERTICAL REACTION (POUNDS)

FIG. 6-2B. WHEEL LOADS (POUNDS), LEFT FRONT, TOTAL - 2 WHEELS
FIG. 6-3. FRONT BOLSTER LATERAL REACTION (POUNDS)
To further correlate with the Stucki data, the IIT model is simulated in a bounce mode in which the car first runs over 2 bumps of amplitude 1 1/2 inch at 60 mph on both tracks and then runs on smooth tracks.

Figures 6-4 and 6-5 show the corresponding loadings on the front and rear center plates respectively as compared to the Stucki data. The general response patterns of the two models are very similar although the maximum center plate loadings on the Stucki model are higher due to a heavier car.

The suspension spring group compressions on such a simulation are represented in Figures 6-6 and 6-7. The corresponding data from the Stucki model was also presented for purpose of comparison. Both models show springs solid, which is a phenomenon actually observed in cases of severe bumps on the rail.

Both center plate loadings and suspension spring compressions on the IIT model are quite similar in amplitudes for the front and rear. This can be explained by the fact that the truck center distance used in the IIT model is 39.5 ft, almost the same as the length of rail joints (39 ft). However, in the case of the Stucki model, a distance of 45 ft was used and this accounts for the general dissimilarities between responses on its front and rear plots.

Table 4 summarizes the comparisons of results between the IIT model and the Stucki model on the bounce mode.

The comparisons show that the two sets of computer output data concerning roll angle, various loadings at the car and truck and spring compressions are extremely close. The Stucki data, however, has been validated against field test data performed on the L & N Railroad Co. rocking test track at Frankfort, Kentucky, on June 24, 1969 (1-10). These comparisons, therefore, show that the current IIT computer model is valid, and based on this, further refinement and development of the mathematical model can continue with confidence. Unfortunately, no test data for freight element response (which may be considered as corresponding to the Stucki data) is currently available to allow total correlation of the motion of a freight element in a box car.
FIG. 6-4. FRONT CENTER PLATE VERTICAL REACTION (LB), BOUNCE MODE - 60 MPH

FIG. 6-5. REAR CENTER PLATE VERTICAL REACTION (LB), BOUNCE MODE - 60 MPH
FIG. 6-6. SPRING GROUP COMPRESSION (IN.) - RIGHT FRONT, BOUNCE MODE - 60 MPH

FIG. 6-7. SPRING GROUP COMPRESSION (IN.) - RIGHT REAR, BOUNCE MODE - 60 MPH
This effect has to be deduced from the accurate modeling of the freight dynamic environment in the box car, and then from laboratory simulation of this environment for individual freight elements. The net response of the Freight Element in the box car is the composite motion derived from these two effects.

Table 4. Comparison of Results with Stucki's - Bounce Mode

<table>
<thead>
<tr>
<th></th>
<th>STUCKI (100-ton hopper car)</th>
<th>IIT (70-ton box car)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Center plate loadings</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. Maximum - front</td>
<td>290 K lb</td>
<td>215 K lb</td>
</tr>
<tr>
<td>b. Maximum - rear</td>
<td>320 K lb</td>
<td>205 K lb</td>
</tr>
<tr>
<td>c. No. of response cycles - front</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II. Suspension spring compressions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. Spring bottoming occurrence - right front</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>b. Interval between spring bottomings - right front</td>
<td>0.45 sec</td>
<td>0.5 sec</td>
</tr>
<tr>
<td></td>
<td>0.45 sec</td>
<td></td>
</tr>
</tbody>
</table>
7. DYNAMICS OF A SINGLE FREIGHT ELEMENT

7.1 Freight Dynamics Study by Computer Model Simulation

Computer models developed by Track Train Dynamics groups throughout the nation and other railroad researchers are primarily designed for studying the dynamics of the car and the truck. As far as these aspects are concerned, we have already been able to generate information from our present computer model. However, our main attention is given to simulate the type of dynamic environment that revenue trains generate in freight cars. Knowing this freight environment as part of a mathematical model, valuable information on the dynamics of the freight can readily be predicted.

Other researchers have made some attempts to study the responses of a freight element under the impact-shock type of excitation (7-1, 1-3). However, the study of the freight dynamics in a 70-ton box car with a typical truck on the road type of environment by computer simulation is believed to be first of its kind.

A freight element located at different places of the box car is likely to be experiencing different acceleration levels. Hence, attempts were made here to consider a fragile freight element (in this case a 150-lb refrigerator) packaged in some cushion material at different locations of the box car and to study the responses of the freight. Some severe over-the-road conditions like the rocking mode and bounce mode were used to excite the model moving at resonant speeds. Study as such will give guidelines to the more severe environment freight responses that the usual revenue train may experience.

7.2 Case Studies

The following cases were simulated for illustration purposes on the freight dynamics:

Case (1) - Rocking mode (17.5 mph), freight element near roof of the box car. The 150-lb freight element is placed at a point 4 ft above the center of gravity of the car body. The lateral accelerations on the freight were plotted as a function of time (Fig. 7-1a). A maximum peak to peak acceleration level
FIG. 7-1A, FREIGHT ELEMENT LATERAL ACCELERATION \((g)\) AT ROOF OF THE CAR, ROCKING MODE - 17.5 MPH

FIG. 7-1B, CAR BODY LATERAL ACCELERATION \((g)\) AT ITS CENTER OF GRAVITY, ROCKING MODE - 17.5 MPH
of 1.5 g is encountered after 8.5 sec of simulation. These accelerations of the freight were compared with the lateral acceleration at the car body center of gravity (Fig. 7-1b) which has a maximum peak to peak value of 1 g. This demonstrates that freight element at locations other than the center of gravity of the car body will sometimes experience more severe levels of accelerations during the rocking mode which is inherited with the conventionally half-staggered rail joint system in North America.

Based on these accelerations, a frequency analysis was made using the program FREQ to identify the most significant freight element vibration frequency, the corresponding Power Spectral Density (PSD) was computed and results plotted in Figures 7-2a and 7-2b. The frequency associated with the highest level of accelerations (.314 g) is around 0.64 Hz at which the PSD value is also a maximum.

Case (2) - Rocking mode (50 mph), freight element near roof of the box car. This is defined as rocking mode simply because of the 1/2 stagger used in the rail joints. The freight element lateral and vertical accelerations versus time are plotted in Figures 7-3a and 7-3b. The lateral peak to peak accelerations on the freight is 0.2 g at a frequency of 2 Hz. The vertical peak to peak accelerations is 0.1 g at 1.25 Hz (Fig. 7-3b).

Case (3) - Bounce mode (50 mph), freight element at front end of the box car (near floor). The freight element is placed at the front end of the car body. Due to the severe pitching motions of the car in the bounce mode, at which the rail joints on opposite tracks are in phase, the accelerations that the freight element is experiencing is higher at this location than that at the center of gravity of the car. Computer simulations were made and the freight vertical accelerations plotted against time in Figure 7-4a. The frequency analysis (Fig. 7-5a) shows that the frequency associated with the highest level of accelerations on the freight (.424 g) is 2 Hz. The PSD plot (Fig. 7-5b) also indicates that most of the vibratory energy is around this particular frequency.
FIG. 7-2A. FREIGHT ELEMENT ACCELERATION (g) SPECTRUM, LATERAL, ROCKING MODE - 17.5 MPH

FIG. 7-2B. PSD ANALYSIS (g²/Hz), FREIGHT ELEMENT LATERAL, ROCKING MODE - 17.5 MPH
FIG. 7-3A. LATERAL ACCELERATION ($g$) OF FREIGHT ELEMENT AT ROOF OF THE CAR, ROCKING MODE - 50 MPH

FIG. 7-3B. VERTICAL ACCELERATION ($g$) OF FREIGHT ELEMENT AT CENTER OF GRAVITY OF CAR BODY, ROCKING MODE - 50 MPH
FIG. 7-4A. VERTICAL ACCELERATION (g) OF FREIGHT ELEMENT AT FRONT END OF CAR, BOUNCE MODE - 50 MPH

FIG. 7-4B. VERTICAL ACCELERATION (g) OF FREIGHT ELEMENT AT CENTER OF GRAVITY OF CAR BODY, BOUNCE MODE - 50 MPH
FIG. 7-5A. FREIGHT ELEMENT ACCELERATION (g) SPECTRUM, VERTICAL BOUNCE MODE - 50 MPH

FIG. 7-5B. PSD ANALYSIS (g²/Hz), FREIGHT ELEMENT VERTICAL, BOUNCE MODE - 50 MPH
Case (4) - Bounce mode (50 mph), freight at center of gravity of the car body. Figure 7-4b shows the freight element vertical accelerations at center of gravity of the car at 50 mph in the bounce mode. A maximum peak to peak accelerations of 1.3 g is predicted as compared to 1.75 g when the freight is at the front end of the car.

The above examples of freight dynamics simulation illustrates some of the bounds on the freight responses when moving over the railroads. Detail study on freight dynamics will be part of the next phase of work.
8. CONCLUSIONS AND RESULTS

1. A mathematical model of the dynamics of a railroad box car carrying a freight element has been developed.

2. It has been shown by comparison with test data, that this model, which represents the car body, freight element, truck bolsters, wheelsets and sideframes, suspension systems and track elasticity and profile as a twenty-seven degree of freedom system, can satisfactorily simulate the dynamics of the typical U.S. box car and freight traveling on tangent track at various operating speeds.

3. Using the computer simulation developed here, it is possible to determine the accelerations, velocities displacements and forces experienced by a freight element, or by the various components of a box car/truck, due to excitation from the rail and/or coupler forces.

4. The output values may be expressed in either the time domain or in the frequency domain, at the option of the user.

5. This study has shown that Coulomb friction damping in a railcar suspension system can be satisfactorily modeled as equivalent viscous damping.

6. This new method facilitates development of new truck designs by making it easier for the designer to investigate the effects of a wide range of configurations and specifications of both friction and hydraulic dampers.

7. When compared to data published by the A. Stucki Company, the simulation developed here shows excellent results for prediction of the box car motions in the Rocking Mode and the Vertical Bounce Mode.

8. The dynamic response of a typical freight element subjected to the vibration environment in a box car under operating conditions can now be predicted by this model.

9. Further pursuit of this freight response study will lead to a thorough understanding of the dynamic response of the freight element. This will indicate possible design modifications in packaging systems, freight car and truck characteristics, etc., which will minimize freight damage due to vibrations.
APPENDIX A

EQUATIONS OF MOTION
The motion of a rigid body in space can be described by rotations and translations. Rotations can be referenced from any inertial coordinates or body axes. However, the angular velocity components about the body axes (which rotates with the body) cannot be integrated to obtain angular displacements about these axes (Chapter 4 Rigid Body Dynamics, Methods of Analytical Dynamics, Leonard Meirovitch). Therefore, it will be unsatisfactory to describe the orientation of a rigid body in space by the body angular velocity components.

One set of independent coordinates which can carry out the transformation from one Cartesian system of axes to another is the Euler's angles. The reason for using Euler's angles is that the three components of the body angular velocity can be expressed in terms of Euler's angles and their time derivatives. Thus, in this analysis, the Euler's angles method is adapted for description of rotations.

The translational coordinates used are inertial coordinates fixed in space, with the coordinate origins at the center of gravity of the various masses when the springs are not extended or compressed, i.e. masses sitting at the free length of the springs. All displacements, velocities and accelerations in translation are referenced from these coordinate origins.

The translation coordinates, together with the Euler's angles constitute the generalized coordinates of the system. In the present model analysis, there are 27 such generalized coordinates.

Once the generalized coordinates are set up, equations of motion can be written for the system by using the method of Lagrange's equation. The Lagrange's equation is an equation of motion in each of the generalized coordinates. It sums up the forces acting on a mass due to the kinetic energy, both rotational and translational, the potential energy associated with spring and gravity, the dissipation energy from damping system and the generalized forces. Details of derivation of each of the energies are discussed later in this section of the appendix.
Figure A-1a shows how Euler's angles provide a description of the body orientation in space. The transformation of body axes x, y, z to one set of inertial Cartesian coordinates X, Y, Z is carried out by three successive rotations:

1. rotate about x axis through an angle \( \phi \) brings x, y, z into x, y', z'
2. rotate about y' axis through an angle \( \psi \) brings x, y', z' into x', y', Z
3. rotate about Z axis through an angle \( \alpha \) brings x', y', z into X, Y, Z.

Since x, y' and Z are the axes of rotation, the body angular velocity components \( \phi \), \( \psi \), and \( \alpha \) are directed along these axes respectively.

Denoting the inertial angular components about X, Y, Z axes as \( \omega_1 \), \( \omega_2 \) and \( \omega_3 \) respectively, it can readily be seen, from Figure A-1b, that the following relationships are true by resolving the body angular velocities \( \phi \), \( \psi \), and \( \alpha \) along:

1. X axis, \( \omega_1 = \dot{\phi} \cos \psi \cos \alpha + \dot{\psi} \sin \alpha \),
2. Y axis, \( \omega_2 = \dot{\psi} \cos \alpha - \dot{\phi} \cos \psi \sin \alpha \), and
3. Z axis, \( \omega_3 = \dot{\phi} \sin \psi + \dot{\alpha} \).

Similarly, writing the above relationships \( \omega_{ij} \) for the five masses, where \( i = 1, \ldots, 5 \) and \( j = 1, 2, 3 \):
FIG. A-1A. EULER’S ANGLES

FIG. A-1B. TRANSFORMATION OF BODY ANGULAR VELOCITIES TO INERTIAL ANGULAR COMPONENTS
a) For the car body:
\[
\begin{align*}
\omega_{11} &= \dot{\phi}_1 \cos \psi_1 \cos \alpha_1 + \dot{\psi}_1 \sin \alpha_1 \\
\omega_{12} &= \dot{\psi}_1 \cos \alpha_1 - \dot{\phi}_1 \cos \psi_1 \sin \alpha_1 \\
\omega_{13} &= \dot{\phi}_1 \sin \psi_1 + \dot{\alpha}_1
\end{align*}
\]

b) For the front bolster:
\[
\begin{align*}
\omega_{21} &= \dot{\phi}_2 \cos \psi_2 \cos \alpha_2 + \dot{\psi}_2 \sin \alpha_2 \\
\omega_{22} &= \dot{\psi}_2 \cos \alpha_2 - \dot{\phi}_2 \cos \psi_2 \sin \alpha_2 \\
\omega_{23} &= \dot{\phi}_2 \sin \psi_2 + \dot{\alpha}_2
\end{align*}
\]

c) For the rear bolster:
\[
\begin{align*}
\omega_{31} &= \dot{\phi}_3 \cos \psi_3 \cos \alpha_3 + \dot{\psi}_3 \sin \alpha_3 \\
\omega_{32} &= \dot{\psi}_3 \cos \alpha_3 - \dot{\phi}_3 \cos \psi_3 \sin \alpha_3 \\
\omega_{33} &= \dot{\phi}_3 \sin \psi_3 + \dot{\alpha}_3
\end{align*}
\]

d) For the front axle-wheel set-sideframe assembly:
\[
\begin{align*}
\omega_{41} &= \dot{\phi}_4 \cos \psi_4 \cos \alpha_4 + \dot{\psi}_4 \sin \alpha_4 \\
\omega_{42} &= \dot{\psi}_4 \cos \alpha_4 - \dot{\phi}_4 \cos \psi_4 \sin \alpha_4 \\
\omega_{43} &= \dot{\phi}_4 \sin \psi_4 + \dot{\alpha}_4
\end{align*}
\]

e) For the rear axle-wheel set-sideframe assembly:
\[
\begin{align*}
\omega_{51} &= \dot{\phi}_5 \cos \psi_5 \cos \alpha_5 + \dot{\psi}_5 \sin \alpha_5 \\
\omega_{52} &= \dot{\psi}_5 \cos \alpha_5 - \dot{\phi}_5 \cos \psi_5 \sin \alpha_5 \\
\omega_{53} &= \dot{\phi}_5 \sin \psi_5 + \dot{\alpha}_5
\end{align*}
\]

where \(\phi\)--angle of pitch, \(\psi\)--angle of roll and \(\alpha\)--angle of yaw. Note: The freight element at this stage is assumed to have only translational degrees of freedom and hence no rotational angles associated with it.

**Kinetic Energy of the System**

The total kinetic energy is the sum of the rotational kinetic energy and translational kinetic energy.
Kinetic Energy of the Car Body. Assuming all rotations about the principal axes allowed for the car body

\[
K.E_{\text{rotational}} = \frac{1}{2}(I_{x_1}\omega_{11}^2 + I_{y_1}\omega_{12}^2 + I_{z_1}\omega_{13}^2)
\]

where \(I_{x_1}, I_{y_1}\) and \(I_{z_1}\) are the principal moments of inertia about \(x_1, y_1, z_1\) axes of the car body.

Assuming the car body can translate in the three principal directions

\[
K.E_{\text{translational}} = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2)
\]

where \(m_1\) = mass of the car body.

Substituting the Euler's angles for \(\omega_{11}, \omega_{12}\) and \(\omega_{13}\), we have, for kinetic energy of the car body

\[
E_1 = \frac{1}{2} I_{x_1} (\dot{\phi}_1 \cos \psi_1 \cos \alpha_1 + \dot{\psi}_1 \sin \alpha_1)^2 + \frac{1}{2} I_{y_1} (\dot{\phi}_1 \cos \psi_1 - \dot{\phi}_1 \cos \psi_1 \sin \alpha_1)^2 + \frac{1}{2} I_{z_1} (\dot{\phi}_1 \sin \psi_1 + \dot{\psi}_1)^2 + \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2)
\]

Kinetic Energy for the Front Bolster. Similarly,

\[
E_2 = \frac{1}{2} (I_{x_2} \omega_{21}^2 + I_{y_2} \omega_{22}^2 + I_{z_2} \omega_{23}^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)
\]

Since we assume the lateral displacements of the bolsters constrained to that of the car body, we have, see Figure A-2,

\[
x_2 = x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1
\]

\[
\dot{x}_2 = \dot{x}_1 - r_1 \cos \psi_1 \dot{\psi}_1 - d_1 \cos \alpha_1 \dot{\alpha}_1
\]

Substituting \(\omega_{21}, \omega_{22}, \omega_{23}\) and \(\dot{x}_2\), we have

\[
E_2 = \frac{1}{2} I_{x_2} (\dot{\phi}_2 \cos \psi_2 \cos \alpha_2 + \dot{\psi}_2 \sin \alpha_2)^2 + \frac{1}{2} I_{y_2} (\dot{\phi}_2 \cos \psi_2 - \dot{\phi}_2 \cos \psi_2 \sin \alpha_2)^2 + \frac{1}{2} I_{z_2} (\dot{\phi}_2 \sin \psi_2 + \dot{\psi}_2)^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + 2 r_1 \dot{x}_1 \cos \psi_1 \dot{\psi}_1 + r_1^2 \cos^2 \psi_1 \dot{\psi}_1^2 - 2 d_1 \dot{x}_1 \cos \alpha_1 \dot{\alpha}_1 + 2 r_1 d_1 \cos \psi_1 \cos \alpha_1 \dot{\psi}_1 \dot{\alpha}_1 + \dot{d}_2 \cos \alpha_1 \dot{\alpha}_1^2) + \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2} m_2 \dot{z}_2^2
\]
Kinetic Energy of the Rear Bolster.

\[
E_3 = \frac{1}{2} I_{x3} (\dot{\phi}_3 \cos \psi_3 \cos \alpha_3 + \dot{\psi}_3 \sin \alpha_3)^2 \\
+ \frac{1}{2} I_{y3} (\dot{\psi}_3 \cos \alpha_3 - \dot{\phi}_3 \cos \psi_3 \sin \alpha_3)^2 \\
+ \frac{1}{2} I_{z3} (\dot{\phi}_3 \sin \psi_3 + \dot{\alpha}_3)^2 + \frac{1}{2} m_3 (\ddot{x}_1^2 - 2r_1 \dot{x}_1 \cos \psi_1 \dot{\psi}_1 \\
+ r_1^2 \cos^2 \psi_1 \dot{\psi}_1^2 + 2d_1 \dot{x}_1 \cos \alpha_1 \dot{\alpha}_1 - 2r_1 d_1 \cos \psi_1 \dot{\psi}_1 \cos \alpha_1 \dot{\alpha}_1 \\
+ d_1^2 \cos^2 \alpha_1 \dot{\alpha}_1^2) + \frac{1}{2} m_3 \dot{y}_3^2 + \frac{1}{2} m_3 \dot{z}_3^2
\]

Kinetic Energy of the Front Axle-Wheel Set-Sideframe Assembly.

\[
E_4 = \frac{1}{2} I_{x4} (\dot{\phi}_4 \cos \psi_4 \cos \alpha_4 + \dot{\psi}_4 \sin \alpha_4)^2 \\
+ \frac{1}{2} I_{y4} (\dot{\psi}_4 \cos \alpha_4 - \dot{\phi}_4 \cos \psi_4 \sin \alpha_4)^2 \\
+ \frac{1}{2} I_{z4} (\dot{\phi}_4 \sin \psi_4 + \dot{\alpha}_4)^2 + \frac{1}{2} m_4 (\ddot{x}_4^2 + \dot{y}_4^2 + \dot{z}_4^2)
\]

Kinetic Energy of the Rear Axle-Wheel Set-Sideframe Assembly.

\[
E_5 = \frac{1}{2} I_{x5} (\dot{\phi}_5 \cos \psi_5 \cos \alpha_5 + \dot{\psi}_5 \sin \alpha_5)^2 \\
+ \frac{1}{2} I_{y5} (\dot{\psi}_5 \cos \alpha_5 - \dot{\phi}_5 \cos \psi_5 \sin \alpha_5)^2 \\
+ \frac{1}{2} I_{z5} (\dot{\phi}_5 \sin \psi_5 + \dot{\alpha}_5)^2 + \frac{1}{2} m_5 (\ddot{x}_5^2 + \dot{y}_5^2 + \dot{z}_5^2)
\]

Kinetic Energy of the Freight Element.

\[
E_6 = \frac{1}{2} m_6 (\ddot{x}_6^2 + \ddot{y}_6^2 + \ddot{z}_6^2)
\]

Therefore, the total kinetic energy of the system is

\[
E = E_1 + E_2 + E_3 + E_4 + E_5 + E_6
\]

Potential Energy of the System

The total potential energy is the sum of the spring potential energy and the gravitational potential energy of the entire system.

In our present mathematical model there are six masses connected by springs and dampers. The corresponding potential energy associated with each group of springs is derived as follows.

Spring Potential Energy Between the Car Body and the Front Truck Bolster. When a spring is displaced distance \( x \) units, the spring potential energy is simply \( \frac{1}{2} Kx^2 \), where \( K \) is the stiffness of the spring.

Referring to Figure A-3, the spring potential energy associated with the vertical springs with stiffness \( K_1 \) and \( K_2 \), respectively are
FIG. A-3. VERTICAL SPRINGS AT CAR BODY AND TRUCK BOLSTER INTERFACE
\[ V_1 = \frac{1}{2} K_1 (z_1 - z_2 - e(\sin \psi_1 - \sin \psi_2) + d_1 \sin \phi_1)^2 \]
\[ V_2 = \frac{1}{2} K_2 (z_1 - z_2 + e(\sin \psi_1 - \sin \psi_2) + d_1 \sin \phi_1)^2 \]

For the two springs modelling the side bearings, i.e. \( K_5 \) and \( K_6 \), we have
\[ V_5 = \frac{1}{2} K_5 \delta_1 (z_1 - z_2 - g_1(\sin \psi_1 - \sin \psi_2) + d_1 \sin \phi_1 + GAP)^2 \]
where \( \delta_1 = 1 \) when
\[ (z_1 - z_2 - g_1(\sin \psi_1 - \sin \psi_2) + d_1 \sin \phi_1 + GAP) < 0 \]
and \( \delta_1 = 0 \) when
\[ (z_1 - z_2 - g_1(\sin \psi_1 - \sin \psi_2) + d_1 \sin \phi_1 + GAP) \geq 0 \]
\[ V_6 = \frac{1}{2} K_6 \delta_2 (z_1 - z_2 + g_1(\sin \psi_1 - \sin \psi_1) + d_1 \sin \phi_1 + GAP)^2 \]
Similarly, \( \delta_2 = 1 \) when
\[ (z_1 - z_2 + g_1(\sin \psi_1 - \sin \psi_2) + d_1 \sin \phi_1 + GAP) < 0 \]
and \( \delta_2 = 0 \) when
\[ (z_1 - z_2 + g_1(\sin \psi_1 - \sin \psi_2) + d_1 \sin \phi_1 + GAP) \geq 0 \]

The function of \( \delta_1 \) and \( \delta_2 \) is to check the side bearing clearance between the car body and truck bolster and if no clearance exists, the corresponding spring is added in parallel to \( K_1 \) and \( K_2 \).

**Spring Potential Energy Between the Car Body and the Rear Truck Bolster.** Similarly we can write
\[ V_7 = \frac{1}{2} K_7 (z_1 - z_3 - e(\sin \psi_1 - \sin \psi_3) - d_1 \sin \phi_1)^2 \]
\[ V_8 = \frac{1}{2} K_8 (z_1 - z_3 + e(\sin \psi_1 - \sin \psi_3) - d_1 \sin \phi_1)^2 \]
\[ V_{11} = \frac{1}{2} K_{11} \delta_3 (z_1 - z_3 - g_1(\sin \psi_1 - \sin \psi_3) - d_1 \sin \phi_1 + GAP)^2 \]
\[ V_{12} = \frac{1}{2} K_{12} \delta_4 (z_1 - z_3 + g_1(\sin \psi_1 - \sin \psi_3) - d_1 \sin \phi_1 + GAP)^2 \]

**Spring Potential Energy for the Front Truck Suspension Springs.** Refer to Figure A-4, we have
\[ V_{13} = \frac{1}{2} K_{13} (z_2 - z_4 - h_2(\sin \psi_2 - \sin \psi_4)) \]
\[ + d_2 \sin \phi_2 - d_4 \sin \phi_4)^2 \]
\[ V_{14} = \frac{1}{2} K_{14} (z_2 - z_4 + h_2(\sin \psi_2 - \sin \psi_4)) \]
\[ + d_2 \sin \phi_2 - d_4 \sin \phi_4)^2 \]
\[ V_{15} = \frac{1}{2} K_{15} (z_2 - z_4 - h_2 (\sin \psi_2 - \sin \psi_4) \]
\[ - (d_2 \sin \phi_2 - d_4 \sin \phi_4))^2 \]
\[ V_{16} = \frac{1}{2} K_{16} (z_2 - z_4 + h_2 (\sin \psi_2 - \sin \psi_4) \]
\[ - (d_2 \sin \phi_2 - d_4 \sin \phi_4))^2 \]
\[ VBOM_9 = \frac{1}{2} KBO_{9} (z_2 - z_4 - h_2 (\sin \psi_2 - \sin \psi_4) + TL)^2 \]
\[ VBOM_{10} = \frac{1}{2} KBO_{10} (z_2 - z_4 + h_2 (\sin \psi_2 - \sin \psi_4) + TL)^2 \]

where \( \delta_9, \delta_{10} = 1 \) or 0 depending on whether suspension springs bottom out or not.

**Spring Potential Energy for the Rear Truck Suspension Springs:**
\[ V_{17} = \frac{1}{2} K_{17} (z_3 - z_5 - h_3 (\sin \psi_3 - \sin \psi_5) \]
\[ + d_3 \sin \phi_3 - d_5 \sin \phi_5)^2 \]
\[ V_{18} = \frac{1}{2} K_{18} (z_3 - z_5 + h_3 (\sin \psi_3 - \sin \psi_5) \]
\[ + d_3 \sin \phi_3 - d_5 \sin \phi_5)^2 \]
\[ V_{19} = \frac{1}{2} K_{19} (z_3 - z_5 - h_3 (\sin \psi_3 - \sin \psi_5) \]
\[ - (d_3 \sin \phi_3 - d_5 \sin \phi_5)^2 \]
\[ V_{20} = \frac{1}{2} K_{20} (z_3 - z_5 + h_3 (\sin \psi_3 - \sin \psi_5) \]
\[ - (d_3 \sin \phi_3 - d_5 \sin \phi_5)^2 \]
\[ VBOM_{11} = \frac{1}{2} KBO_{11} (z_3 - z_5 - h_3 (\sin \psi_3 - \sin \psi_5) + TL)^2 \]
\[ VBOM_{12} = \frac{1}{2} KBO_{12} (z_3 - z_5 + h_3 (\sin \psi_3 - \sin \psi_5) + TL)^2 \]

where \( \delta_{11} , \delta_{12} = 1 \) or 0 depending on whether rear suspension groups bottom out or not.

**Spring Potential Energy for Lateral Springs Between Front Bolster and Sideframe.** Refer to Figure A-5,
\[ V_{13L} = \frac{1}{2} K_{13L} (x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1 - x_4 - (d_2 \sin \alpha_2 \]
\[ - d_4 \sin \alpha_4) - (r_2 \sin \psi_2 + r_4 \sin \psi_4))^2 \]
\[ V_{14L} = \frac{1}{2} K_{14L} (x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1 - x_4 - (d_2 \sin \alpha_2 \]
\[ - d_4 \sin \alpha_4) - (r_2 \sin \psi_2 + r_4 \sin \psi_4))^2 \]
FIG. A-4. VERTICAL SPRINGS AT BOLSTER AND TRUCK INTERFACE

FIG. A-5. LATERAL SPRINGS BETWEEN FRONT BOLSTER AND TRUCK
Potential Energy for the Lateral Springs Between Rear Bolster and Sideframe.

\[
V_{1sL} = \frac{1}{2} K_{1sL} (x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1 - x_5 + d_2 \sin \alpha_2
- d_4 \sin \alpha_4 - (r_2 \sin \psi_2 + r_4 \sin \psi_4))^2
\]

\[
V_{16L} = \frac{1}{2} K_{1sL} (x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1 - x_5 + d_2 \sin \alpha_2
- d_4 \sin \alpha_4 - (r_2 \sin \psi_2 + r_4 \sin \psi_4))^2
\]

\[
VGIB_5 = \frac{1}{2} KGIB_5 (x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1 - x_4 - GIB)^2
\]

\[
VGIB_6 = \frac{1}{2} KGIB_6 (x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1 - x_4 + GIB)^2
\]

where \( \delta_5 \) and \( \delta_6 \) are 1 or 0 depending on the GIB clearance between the bolster and the column of the sideframe.

Potential Energy for the Track Springs. Refer to Figure A-6 (for front truck),

**Vertical Springs.**

\[
V_{21} = \frac{1}{2} K_{21} (z_4 - H \sin \psi_4 + D \sin \phi_4 - V_1)^2
\]

\[
V_{22} = \frac{1}{2} K_{22} (z_4 + H \sin \psi_4 + D \sin \phi_4 - V_2)^2
\]

\[
V_{23} = \frac{1}{2} K_{23} (z_4 - H \sin \psi_4 - D \sin \phi_4 - V_3)^2
\]

\[
V_{24} = \frac{1}{2} K_{24} (z_4 + H \sin \psi_4 - D \sin \phi_4 - V_4)^2
\]

**Lateral Springs.**

\[
V_{21L} = \frac{1}{2} K_{21L} (x_4 - D \sin \alpha_4 - R \sin \psi_4 + R_5)^2
\]
FIG. A-6. TRACK SPRINGS AT FRONT TRUCK
\[ V_{22L} = \frac{1}{2} K_{22L}(x_4 - D \sin \alpha_4 - R \sin \psi_4 - R_{10})^2 \]
\[ V_{23L} = \frac{1}{2} K_{23L}(x_4 + D \sin \alpha_4 - R \sin \psi_4 + R_{11})^2 \]
\[ V_{24L} = \frac{1}{2} K_{24L}(x_4 + D \sin \alpha_4 - R \sin \psi_4 - R_{12})^2 \]

Similarly, for rear truck,
\[ V_{25} = \frac{1}{2} K_{25}(z_5 - H \sin \psi_5 + D \sin \phi_5 - V_5)^2 \]
\[ V_{26} = \frac{1}{2} K_{26}(z_5 + H \sin \psi_5 + D \sin \phi_5 - V_6)^2 \]
\[ V_{27} = \frac{1}{2} K_{27}(z_5 - H \sin \psi_5 - D \sin \phi_5 - V_7)^2 \]
\[ V_{28} = \frac{1}{2} K_{28}(z_5 + H \sin \psi_5 - D \sin \phi_5 - V_8)^2 \]
\[ V_{25L} = \frac{1}{2} K_{25L}(x_5 - D \sin \alpha_5 - R \sin \psi_5 + R_{13})^2 \]
\[ V_{26L} = \frac{1}{2} K_{26L}(x_5 - D \sin \alpha_5 - R \sin \psi_5 - R_{14})^2 \]
\[ V_{27L} = \frac{1}{2} K_{27L}(x_5 + D \sin \alpha_5 - R \sin \psi_5 + R_{15})^2 \]
\[ V_{28L} = \frac{1}{2} K_{28L}(x_5 + D \sin \alpha_5 - R \sin \psi_5 - R_{16})^2 \]

**Potential Energy for the Torsional and Pitching Springs.**
\[ VT_{24} = \frac{1}{2} K_{T24}(\alpha_2 - \alpha_4)^2 \]
\[ VT_{35} = \frac{1}{2} K_{T35}(\alpha_3 - \alpha_5)^2 \]
\[ VP_{24} = \frac{1}{2} K_{P24}(\phi_2 - \phi_4)^2 \]
\[ VP_{35} = \frac{1}{2} K_{P35}(\phi_3 - \phi_5)^2 \]

**Potential Energy for Springs Between Freight Element and the C.G. of Car Body.**
\[ V_F = \frac{1}{2} K_F(x_6 - x_1)^2 + \frac{1}{2} K_F(y_6 - y_1)^2 + \frac{1}{2} K_F(z_6 - z_1)^2 \]

**Gravitational Potential Energy of the System.**
\[ V_G = m_1g z_1 + m_2g z_2 + m_3g z_3 + m_4g z_4 + m_5g z_5 + m_6g z_6 \]
Total Potential Energy of the System.

\[ V = V_1 + V_2 + V_5 + V_6 + V_7 + V_8 + V_{11} + V_{12} + V_{13} + V_{14} + V_{15} + V_{16} + V_{BOM9} + V_{BOM10} + V_{17} + V_{18} + V_{19} + V_{20} + V_{BOM11} + V_{BOM12} + V_{13L} + V_{14L} + V_{15L} + V_{16L} + V_{GIB5} + V_{GIB6} + V_{17L} + V_{18L} + V_{19L} + V_{20L} + V_{GIB7} + V_{GIB8} + V_{21} + V_{22} + V_{23} + V_{24} + V_{21L} + V_{22L} + V_{23L} + V_{24L} + V_{25} + V_{26} + V_{27} + V_{25L} + V_{26L} + V_{27L} + V_{28L} + V_{VT24} + V_{VT35} + V_{P24} + V_{P35} + V_F + V_G \]

Dissipation Energy

Energy is dissipated through the various dampers of the system. If treated as viscous dampers, the damping force is proportional to the velocity of the damping action, i.e. of the form \( cx \) and the energy dissipated is of the form \( \frac{1}{2} cx^2 \). The dissipation energy of the system is

\[ D = \frac{1}{2} C_{13} (\dot{z}_2 - \dot{z}_4 - h_2 (\cos \psi_2 \dot{\psi}_2 - \cos \psi_4 \dot{\psi}_4) + d_2 \cos \phi_2 \dot{\phi}_2 - d_4 \cos \phi_4 \dot{\phi}_4)^2 + \frac{1}{2} C_{14} (\dot{z}_2 - \dot{z}_4 - h_2 (\cos \psi_2 \dot{\psi}_2 - \cos \psi_4 \dot{\psi}_4) + d_2 \cos \phi_2 \dot{\phi}_2 - d_4 \cos \phi_4 \dot{\phi}_4)^2 + \frac{1}{2} C_{15} (\dot{z}_2 - \dot{z}_4 - h_2 (\cos \psi_2 \dot{\psi}_2 - \cos \psi_4 \dot{\psi}_4) + d_2 \cos \phi_2 \dot{\phi}_2 - d_4 \cos \phi_4 \dot{\phi}_4)^2 + \frac{1}{2} C_{16} (\dot{z}_2 - \dot{z}_4 - h_2 (\cos \psi_2 \dot{\psi}_2 - \cos \psi_4 \dot{\psi}_4) + d_2 \cos \phi_2 \dot{\phi}_2 - d_4 \cos \phi_4 \dot{\phi}_4)^2 + \frac{1}{2} C_{17} (\dot{z}_3 - \dot{z}_5 - h_3 (\cos \psi_3 \dot{\psi}_3 - \cos \psi_5 \dot{\psi}_5) + d_3 \cos \phi_3 \dot{\phi}_3 - d_5 \cos \phi_5 \dot{\phi}_5)^2 + \frac{1}{2} C_{18} (\dot{z}_3 - \dot{z}_5 - h_3 (\cos \psi_3 \dot{\psi}_3 - \cos \psi_5 \dot{\psi}_5) + d_3 \cos \phi_3 \dot{\phi}_3 - d_5 \cos \phi_5 \dot{\phi}_5)^2 + \frac{1}{2} C_{19} (\dot{z}_3 - \dot{z}_5 - h_3 (\cos \psi_3 \dot{\psi}_3 - \cos \psi_5 \dot{\psi}_5) + d_3 \cos \phi_3 \dot{\phi}_3 - d_5 \cos \phi_5 \dot{\phi}_5)^2 + \frac{1}{2} C_{20} (\dot{z}_3 - \dot{z}_5 - h_3 (\cos \psi_3 \dot{\psi}_3 - \cos \psi_5 \dot{\psi}_5) + d_3 \cos \phi_3 \dot{\phi}_3 - d_5 \cos \phi_5 \dot{\phi}_5)^2 + \frac{1}{2} (C_{13L} + C_{14L}) (\dot{x}_1 - r_1 \cos \psi_1 \dot{\psi}_1 - d_1 \cos \alpha_1 \dot{\alpha}_1 - \dot{x}_4 - (d_2 \cos \alpha_2 \dot{\alpha}_2 - d_4 \cos \alpha_4 \dot{\alpha}_4) - (r_2 \cos \psi_2 \dot{\psi}_2 + r_4 \cos \psi_4 \dot{\psi}_4))^2 + \frac{1}{2} (C_{15L} + C_{16L}) (\dot{x}_1 - r_1 \cos \psi_1 \dot{\psi}_1 - d_1 \cos \alpha_1 \dot{\alpha}_1 - \dot{x}_4 + (d_2 \cos \alpha_2 \dot{\alpha}_2 - d_4 \cos \alpha_4 \dot{\alpha}_4) - (r_2 \cos \psi_2 \dot{\psi}_2 + r_4 \cos \psi_4 \dot{\psi}_4))^2 + \frac{1}{2} (C_{17L} + C_{18L}) (\dot{x}_1 - r_1 \cos \psi_1 \dot{\psi}_1 + d_1 \cos \alpha_1 \dot{\alpha}_1 - \dot{x}_5 + (d_2 \cos \alpha_2 \dot{\alpha}_2 - d_4 \cos \alpha_4 \dot{\alpha}_4) - (r_2 \cos \psi_2 \dot{\psi}_2 + r_4 \cos \psi_4 \dot{\psi}_4))^2 + \frac{1}{2} (C_{19L} + C_{20L}) (\dot{x}_1 - r_1 \cos \psi_1 \dot{\psi}_1 + d_1 \cos \alpha_1 \dot{\alpha}_1 - \dot{x}_5 + (d_2 \cos \alpha_2 \dot{\alpha}_2 - d_4 \cos \alpha_4 \dot{\alpha}_4) - (r_2 \cos \psi_2 \dot{\psi}_2 + r_4 \cos \psi_4 \dot{\psi}_4))^2 \]
\[
- (d_3 \cos \alpha_3 \dot{\alpha}_3 - d_5 \cos \alpha_5 \dot{\alpha}_5) - (r_3 \cos \psi_3 \dot{\psi}_3 + r_5 \cos \psi_5 \dot{\psi}_5))^2 \\
+ \frac{1}{2} (C_{15}L + C_{20}L)(\dot{x}_1 - r_1 \cos \psi_1 \dot{\psi}_1 + d_1 \cos \alpha_1 \dot{\alpha}_1 - \dot{x}_5 \\
+ (d_3 \cos \alpha_3 \dot{\alpha}_3 - d_5 \cos \alpha_5 \dot{\alpha}_5) - (r_3 \cos \psi_3 \dot{\psi}_3 + r_5 \cos \psi_5 \dot{\psi}_5))^2 \\
+ \frac{1}{2} CF(\dot{x}_6 - \dot{x}_1)^2 + \frac{1}{2} CF(\dot{y}_6 - \dot{y}_1)^2 + \frac{1}{2} CF(\dot{z}_6 - \dot{z}_1)^2 
\]

Lagrange's equation can have the form

\[
\frac{d}{dt} \left( \frac{\partial E}{\partial \dot{q}_i} \right) - \frac{\partial E}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} = Q_i 
\]

After some algebraic manipulation on each of the generalized coordinates and making the following substitutions, we can then write 27 equations of motion.

SAA = \( z_1 - z_2 \)

SAB = \( \sin \psi_1 - \sin \psi_2 \)

SAC = \( z_1 - z_3 \)

SAD = \( \sin \psi_1 - \sin \psi_3 \)

SAF = \( x_1 - r_1 \sin \psi_1 - d_1 \sin \alpha_1 - x_4 \)

SAI = \( d_1 \sin \phi_1 \)

SAM = \( z_2 - z_4 \)

SAN = \( z_2 - z_4 \)

SAO = \( \sin \psi_2 - \sin \psi_4 \)

SAP = \( \cos \psi_2 \dot{\psi}_2 - \cos \psi_4 \dot{\psi}_4 \)

SAQ = \( d_2 \sin \phi_2 - d_4 \sin \phi_4 \)

SAR = \( d_2 \cos \phi_2 \dot{\phi}_2 - d_4 \cos \phi_4 \dot{\phi}_4 \)

SAU = \( x_1 - r_1 \sin \psi_1 + d_1 \sin \alpha_1 \)

SBA = \( z_3 - z_5 \)

SBB = \( \sin \psi_3 - \sin \psi_5 \)

SBC = \( d_3 \sin \phi_3 - d_5 \sin \phi_5 \)

SBD = \( \dot{z}_3 - \dot{z}_5 \)

SBE = \( \cos \psi_3 \dot{\psi}_3 - \cos \psi_5 \dot{\psi}_5 \)
These are the 27 equations of motion in the generalized coordinate of the system:
For $q_1 = x_1$

$$(m_1 + 2m_2)\ddot{x}_1 + 2m_2r_1 \sin \psi_1 \ddot{\psi}_1 = 2m_2r_1 \cos \psi_1 \ddot{\psi}_1 + (K_{13}L + K_{14}L)SC$$(64) 

+ $(K_{15}L + K_{16}L)SCF + (K_{17}L + K_{18}L)SCH + (K_{19}L + K_{20}L)SCD + (C_{14}L)SCA + (C_{15}L + C_{16}L)SCB + (C_{17}L + C_{18}L)SCC + (C_{19}L + C_{20}L)SCD$

$- CF(x_6 - x_1) + KGIB\delta_5(SAF - GIB) + KGIB\delta_6(SAF + GIB) + KGIB\delta_7(SAU - GIB) + KGIB\delta_8(SAU + GIB) - KF(x_6 - x_1) = 0$

For $q_2 = y_1$

$m_1\ddot{y}_1 + KC(y_1 - y_s) + CC(\dot{y_1} - \dot{y}_s) = 0$

For $q_3 = z_1$

$m_1\ddot{z}_1 + K_1(SAA - SAB + SAI) + K_2(SAA + SAB + SAI) + K_5\delta_1(SAA + g_1SAB + SAI + GAP) + K_6\delta_2(SAA + g_1SAB + SAI + GAP) + K_7(SAC - eSAD - SAI) + K_8(SC - eSAD - SAI) + K_9(SAC + g_1SAD - SAI + GAP) + K_{10}(SAC + g_1SAD - SAI + GAP) + m_1g - KF(z_6 - z_1)$

$- CF(z_6 - z_1) = 0$

For $q_4 = \psi_1$

$(-2m_2r_1)\ddot{x}_1 + (I_x \sin^2 \alpha_1 + I_y \cos \alpha_1 \sin \alpha_1 \dot{\phi}_1 \cos \psi_1 \cos \alpha_1 + \dot{\psi}_1 \sin \alpha_1)$

+ $I_x \cos \alpha_1 \dot{\phi}_1 \cos \psi_1 \cos \alpha_1 + \dot{\psi}_1 \sin \alpha_1$

+ $I_y \sin \alpha_1(- \dot{\phi}_1 \cos \alpha_1 \sin \psi_1 \dot{\psi}_1 - \dot{\phi}_1 \cos \psi_1 \sin \alpha_1 \dot{\phi}_1)$

+ $\psi_1 \cos \alpha_1 \dot{\phi}_1 \dot{\psi}_1(- I_y \sin \alpha_1 \dot{\phi}_1 \cos \psi_1 \sin \alpha_1 \dot{\phi}_1)$

+ $I_y \dot{\psi}_1 \cos \alpha_1(- \dot{\psi}_1 \sin \alpha_1 \dot{\phi}_1 \sin \alpha_1 \sin \psi_1 \dot{\psi}_1)$

- $\dot{\phi}_1 \cos \psi_1 \cos \alpha_1 \dot{\phi}_1 \dot{\psi}_1 + I_x \dot{\phi}_1 \cos \alpha_1 \sin \psi_1(\dot{\phi}_1 \cos \psi_1 \cos \alpha_1)$

+ $\dot{\psi}_1 \sin \alpha_1 - I_y \dot{\phi}_1 \sin \alpha_1 \sin \psi_1(\dot{\psi}_1 \cos \alpha_1 - \dot{\phi}_1 \cos \psi_1 \sin \alpha_1)$

- $I_{z_1} \dot{\phi}_1 \cos \psi_1(\dot{\phi}_1 \sin \psi_1 + \dot{\psi}_1)$

$- K_{2e} \cos \psi_1(SAA + eSAB + SAI) + K_{2e} \cos \psi_1(SAA + eSAB + SAI) - K_5\delta_1 g_1 \cos \psi_1(SAA + g_1SAB + SAI + GAP) + K_6\delta_2 g_1 \cos \psi_1(SAA + g_1SAB + SAI + GAP) - K_7e \cos \psi_1(SAC - eSAD - SAI) + K_8e \cos \psi_1(SAC + eSAD - SAI) - K_{11}\delta_3 g_1 \cos \psi_1(SAC - g_1SAD - SAI + GAP) + K_{12}\delta_4 g_1 \cos \psi_1(SAC + g_1SAD - SAI + GAP)$
\[ (K_{13}L + K_{14}L)r_1 \cos \psi_{1SCE} - (K_{15}L + K_{16}L)r_1 \cos \psi_{1SCF} - (K_{17}L + K_{18}L)r_1 \cos \psi_{1SCH} \\
+ K_{19}L)r_1 \cos \psi_{1SCG} - (K_{19}L + K_{20}L)r_1 \cos \psi_{1SCB} \\
- KGIB\delta_5 r_1 \cos \psi_{1(SAF - GIB)} - KGIB\delta_6 r_1 \cos \psi_{1(SAF + GIB)} \\
- KGIB\delta_7 r_1 \cos \psi_{1(SAU - GIB)} - KGIB\delta_8 r_1 \cos \psi_{1(SAU + GIB)} \\
- (C_{13}L + C_{14}L)r_1 \cos \psi_{1SCA} - (C_{15}L + C_{16}L)r_1 \cos \psi_{1SCB} \\
- (C_{17}L + C_{18}L)r_1 \cos \psi_{1SCC} - (C_{19}L + C_{20}L)r_1 \cos \psi_{1SCD} = 0 \\
\]

For \( q_5 = \phi_1 \)
\[
(I_{x_1} - I_{y_1}) \sin \alpha_1 \ddot{\psi}_1 + (I_{x_1} + I_{y_1} \sin^2 \alpha_1 + I_{z_1} \sin^2 \psi_1) \dot{\phi}_1 \\
+ I_{z_1} \sin \psi_1 \ddot{a}_1 - I_{x_1} \cos \alpha_1 \sin \psi_1 \dot{\psi}_1 (\dot{\phi}_1 \cos \psi_1 \cos \alpha_1 + \dot{\psi}_1 \sin \alpha_1) \\
- I_{x_1} \cos \psi_1 \sin \alpha_1 \dot{a}_1 (\dot{\phi}_1 \cos \psi_1 \cos \alpha_1 + \dot{\psi}_1 \sin \alpha_1) + \\
+ I_{x_1} \cos \psi_1 \cos \alpha_1 (- \dot{\phi}_1 \cos \alpha_1 \sin \psi_1 \dot{\psi}_1 - \dot{\psi}_1 \cos \psi_1 \sin \alpha_1 \ddot{a}_1 \\
+ \psi_1 \cos \alpha_1 \dot{\alpha}_1) + I_{y_1} \sin \alpha_1 \sin \psi_1 \dot{\psi}_1 (\dot{\phi}_1 \cos \psi_1 \alpha_1 - \dot{\phi}_1 \cos \psi_1 \sin \alpha_1) \\
- I_{y_1} \cos \psi_1 \cos \alpha_1 \dot{\alpha}_1 (\dot{\phi}_1 \cos \psi_1 \alpha_1 - \dot{\phi}_1 \cos \psi_1 \sin \alpha_1) \\
+ I_{y_1} \cos \psi_1 \sin \alpha_1 (- \dot{\phi}_1 \sin \alpha_1 \dot{a}_1 + \dot{\phi}_1 \sin \alpha_1 \sin \psi_1 \dot{\psi}_1 \\
- \dot{\phi}_1 \cos \psi_1 \cos \alpha_1 \ddot{a}_1) + I_{z_1} \cos \psi_1 \dot{\psi}_1 (\dot{\phi}_1 \sin \psi_1 + \dot{a}_1) \\
+ I_{z_1} \sin \psi_1 \dot{\phi}_1 \cos \psi_1 \dot{\psi}_1 + K_{1d_1} \cos \phi_1 (SAA - eSAB + SAI) \\
+ K_{2d_1} \cos \phi_1 (SAA + eSAB + SAI) + K_{3\delta_1 d_1} \cos \phi_1 (SAA - g_1 SAB + SAI + GAP) + K_{5\delta_2 d_1} \cos \phi_1 (SAA + g_1 SAB + SAI + GAP) - K_{7d_1} \cos \phi_1 (SAC - eSAD - d_1 \sin \phi_1) - K_{8d_1} \cos \phi_1 (SAC + eSAD - SAI) \\
- K_{11\delta_3 d_1} \cos \phi_1 (SAC - g_1 SAD - SAI + GAP) - K_{12\delta_4 d_1} \cos \phi_1 (SAC + g_1 SAD - SAI + GAP) = 0 \\
\]

For \( q_6 = \alpha_1 \)
\[
I_{z_1} \sin \psi_1 \ddot{\phi}_1 + I_{z_1} + 2m_2 d_1 \cos^2 \alpha_1 + I_{z_1} \cos \psi_1 \ddot{\psi}_1 \\
- 4m_2 d_1 \cos \alpha_1 \sin \alpha_1 \ddot{a}_1 - I_{x_1} (- \dot{\phi}_1 \cos \psi_1 \sin \alpha_1 + \dot{\psi}_1 \cos \alpha_1) \\
(\dot{\phi}_1 \cos \psi_1 \cos \alpha_1 + \dot{\psi}_1 \sin \alpha_1) - I_{y_1} (- \dot{\phi}_1 \sin \alpha_1 - \dot{\psi}_1 \cos \psi_1 \cos \alpha_1) \\
(\dot{\psi}_1 \cos \alpha_1 - \dot{\phi}_1 \cos \psi_1 \sin \alpha_1) + 2m_2 d_1 \cos \alpha_1 \sin \alpha_1 \ddot{a}_1 - (K_{13}L \\
+ K_{14}L) \dot{d}_1 \cos \alpha_1 \dot{SCE} - (K_{15}L + K_{16}L) \dot{d}_1 \cos \alpha_1 \dot{SCF} + (K_{17}L + \\
+ K_{18}L) \dot{d}_1 \cos \alpha_1 \dot{SCG} + (K_{19}L + K_{20}L) \dot{d}_1 \cos \alpha_1 \dot{SCH} - (C_{13}L \\
+ C_{14}L) \dot{d}_1 \cos \alpha_1 \dot{SCA} - (C_{15}L + C_{16}L) \dot{d}_1 \cos \alpha_1 \dot{SCB} \\
- (C_{17}L + C_{18}L) \dot{d}_1 \cos \alpha_1 \dot{SCC} - (C_{19}L + C_{20}L) \dot{d}_1 \cos \alpha_1 \dot{SCD} = 0 \\
\]
\[ + C_1 d_1 \cos \alpha_1 \sigma \ - (C_1 d + C_2 d)d_1 \cos \alpha_1 \sigma \ + (C_2 d \]
\[ + C_1 d_1 \cos \alpha_1 \sigma \ + (C_1 d + C_2 d)d_1 \cos \alpha_1 \sigma \]
\[- K_G d_1 \cos \alpha_1 \sigma \ - K_G d_1 \cos \alpha_1 \sigma \ + K_G d_1 \cos \alpha_1 \sigma \]
\[+ K_G d_1 \cos \alpha_1 \sigma \ + K_G d_1 \cos \alpha_1 \sigma \ = 0 \]

For \( g_1 = z_2 \)
\[ m_2 \dot{x}_2 - K_1 (S_A + eS + S_A) - K_2 (S_A + eS + S_A) - K_3 \]
\[- g_1 (S + eS + S + G) - K_4 (S + eS + S + G) + K_5 \]
\[+ h_2 S + S_A) + K_6 (S + h_2 S + S_A) + K_7 (S + h_2 S + S_A) \]
\[+ K_8 (S + h_2 S + S_A) + m_2 g + C_1 (S + h_2 S + S_A) + C_2 \]
\[+ h_2 S + S_A) + C_3 (S + h_2 S + S_A) + C_4 (S + h_2 S + S_A) \]
\[+ K_B d_1 (S_A - h_2 S + S_A) + K_B d_1 (S_A - h_2 S + S_A) = 0 \]

For \( g_2 = \psi_2 \)
\[ (I_x \sin^2 \alpha + I_y \sin \alpha_2 \dot{\psi}_2 + (I_x - I_y) \sin \alpha_2 \dot{\psi}_2 + I_x \cos \alpha_2 \dot{\psi}_2 \]
\[\dot{\psi}_2 \cos \psi_2 \cos \alpha + \dot{\psi}_2 \sin \alpha \] + \[ I_x \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \] - \[ I_y \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \]
\[- \dot{\psi}_2 \cos \psi_2 \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \] - \[ I_x \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \]
\[- \dot{\psi}_2 \cos \psi_2 \cos \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \]
\[+ \dot{\psi}_2 \sin \alpha_2 \] - \[ I_y \dot{\psi}_2 \sin \psi_2 \sin \alpha_2 \] - \[ I_x \dot{\psi}_2 \cos \psi_2 \sin \alpha_2 \]
\[- \dot{I}_x \dot{\psi}_2 \cos \psi_2 \sin \alpha_2 \] - \[ K_1 (S_A - eS + S_A) \]
\[- K_2 (S_A - eS + S_A) \]
\[+ K_3 (S_A - eS + S_A) \]
\[- K_4 (S_A - eS + S_A) \]
\[+ K_5 \dot{\psi}_2 \cos \psi_2 (\dot{\psi}_2 \sin \psi_2 + \dot{\psi}_2 \sin \psi_2) \]
\[+ K_6 \dot{\psi}_2 \cos \psi_2 (\dot{\psi}_2 \sin \psi_2 + \dot{\psi}_2 \sin \psi_2) \]
\[- K_1 (S_A + eS + S_A) \]
\[- K_2 (S_A + eS + S_A) \]
\[+ K_3 (S_A + eS + S_A) \]
\[- K_4 (S_A + eS + S_A) \]
\[+ K_5 \dot{\psi}_2 \cos \psi_2 (\dot{\psi}_2 \sin \psi_2 + \dot{\psi}_2 \sin \psi_2) \]
\[+ K_6 \dot{\psi}_2 \cos \psi_2 (\dot{\psi}_2 \sin \psi_2 + \dot{\psi}_2 \sin \psi_2) \]

For \( g_3 = \psi_3 \)
\[ (I_x \sin^2 \alpha + I_y \sin \alpha_2 \dot{\psi}_2 + (I_x - I_y) \sin \alpha_2 \dot{\psi}_2 + I_x \cos \alpha_2 \dot{\psi}_2 \]
\[\dot{\psi}_2 \cos \psi_2 \cos \alpha + \dot{\psi}_2 \sin \alpha \] + \[ I_x \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \] - \[ I_y \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \]
\[- \dot{\psi}_2 \cos \psi_2 \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \] - \[ I_x \sin \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \]
\[- \dot{\psi}_2 \cos \psi_2 \cos \alpha \dot{\alpha}_2 + \dot{\psi}_2 \cos \alpha \dot{\alpha}_2 \]
\[+ \dot{\psi}_2 \sin \alpha_2 \] - \[ I_y \dot{\psi}_2 \sin \psi_2 \sin \alpha_2 \] - \[ I_x \dot{\psi}_2 \cos \psi_2 \sin \alpha_2 \]
\[- \dot{I}_x \dot{\psi}_2 \cos \psi_2 \sin \alpha_2 \] - \[ K_1 (S_A + eS + S_A) \]
\[- K_2 (S_A + eS + S_A) \]
[...]

For \( g_7 = z_2 \)
\[ m_2 \dot{x}_2 - K_1 (S_A - eS + S_A) - K_2 (S_A - eS + S_A) - K_3 \]
\[- g_1 (S + eS + S + G) - K_4 (S + eS + S + G) + K_5 \]
\[+ h_2 S + S_A) + K_6 (S + h_2 S + S_A) + K_7 (S + h_2 S + S_A) \]
\[+ K_8 (S + h_2 S + S_A) + m_2 g + C_1 (S + h_2 S + S_A) + C_2 \]
\[+ h_2 S + S_A) + C_3 (S + h_2 S + S_A) + C_4 (S + h_2 S + S_A) \]
\[+ K_B d_1 (S_A - h_2 S + S_A) + K_B d_1 (S_A - h_2 S + S_A) = 0 \]

For \( g_8 = \psi_2 \)
...
\[ + h_2 S A P + S A R) - C_{15} h_2 \cos \psi_2 (S A N - h_2 S A P - S A R) \]
\[ + C_{16} h_2 \cos \psi_2 (S A N + h_2 S A P - S A R) - (C_{13} L + C_{14} L) r_2 \cos \psi_2 S C A \]
\[ - (C_{15} L + C_{16} L) r_2 \cos \psi_2 S C B = 0 \]

For \( q_9 = \phi \)

\[ (I_x - I_y) \sin \alpha_2 \psi_2 + (I_x + I_y \sin^2 \alpha_2 + I_z \sin^2 \psi_2) \phi_2 \]
\[ + I_x \sin \psi_2 \dot{u}_2 - I_x \cos \alpha_2 \sin \psi_2 \dot{u}_2 (\dot{\phi}_2 \cos \psi_2 \cos \alpha_2 + \dot{\psi}_2 \sin \alpha_2) \]
\[ - I_x \cos \psi_2 \sin \alpha_2 \dot{\alpha}_2 (\dot{\phi}_2 \cos \psi_2 \cos \alpha_2 + \dot{\psi}_2 \sin \alpha_2) \]
\[ + I_x \cos \psi_2 \cos \alpha_2 (-\dot{\phi}_2 \cos \alpha_2 \sin \psi_2 \dot{\psi}_2 - \dot{\phi}_2 \cos \psi_2 \sin \alpha_2 \dot{\alpha}_2 \]
\[ + \dot{\psi}_2 \cos \alpha_2 \dot{\alpha}_2 + I_y \sin \alpha_2 \sin \psi_2 \dot{\psi}_2 (\dot{\psi}_2 \cos \alpha_2 - \dot{\phi}_2 \cos \psi_2 \sin \alpha_2) \]
\[ - I_y \cos \psi_2 \cos \alpha_2 \dot{\alpha}_2 (\dot{\psi}_2 \cos \alpha_2 - \dot{\phi}_2 \cos \psi_2 \sin \alpha_2) \]
\[ - I_y \cos \psi_2 \sin \alpha_2 (-\dot{\psi}_2 \sin \alpha_2 \dot{\alpha}_2 + \dot{\phi}_2 \sin \alpha_2 \sin \psi_2 \dot{\psi}_2 \]
\[ - \dot{\phi}_2 \cos \psi_2 \cos \alpha_2 \dot{\alpha}_2 + I_z \cos \psi_2 \dot{\psi}_2 (\dot{\psi}_2 \sin \psi_2 + \dot{\alpha}_2) \]
\[ + I_z \sin \psi_2 \dot{\phi}_2 \cos \psi_2 \dot{\psi}_2 + K_{13} d_2 \cos \phi_2 (S A M - h_2 S A O + S A Q) \]
\[ + K_{14} d_2 \cos \phi_2 (S A M + h_2 S A O - S A Q) - K_{15} d_2 \cos \phi_2 (S A M - h_2 S A O - S A Q) \]
\[ - K_{16} d_2 \cos \phi_2 (S A M + h_2 S A O - S A Q) + K P_{24} (\phi_2 - \phi_4) \]
\[ + C_{13} d_2 \cos \phi_2 (S A N - h_2 S A P + S A R) + C_{14} d_2 \cos \phi_2 (S A N + h_2 S A P + S A R) \]
\[ - C_{15} d_2 \cos \phi_2 (S A N - h_2 S A P - S A R) - C_{16} d_2 \cos \phi_2 (S A N + h_2 S A P \]
\[ - S A R) = 0 \]

For \( q_{10} = \alpha \)

\[ I_z (\dot{\phi}_2 \sin \psi_2 + \dot{\phi}_2 \cos \psi_2 \dot{\psi}_2 + \dot{\alpha}_2) - I_x(- \dot{\phi}_2 \cos \psi_2 \sin \alpha_2 \]
\[ + \dot{\psi}_2 \cos \alpha_2) (\dot{\phi}_2 \cos \psi_2 \cos \alpha_2 + \dot{\psi}_2 \sin \alpha_2) - I_y(- \dot{\psi}_2 \sin \alpha_2 \]
\[ - \dot{\phi}_2 \cos \psi_2 \cos \alpha_2) (\dot{\psi}_2 \cos \alpha_2 - \dot{\phi}_2 \cos \psi_2 \sin \alpha_2) - (K_{13} L \]
\[ + K_{14} L)d_2 \cos \alpha_2 S C E + (K_{15} L + K_{16} L) d_2 \cos \alpha_2 S C F + K T_{24} (\alpha_2 - \alpha_4) \]
\[ - (C_{13} L + C_{14} L)d_2 \cos \alpha_2 S C A + (C_{15} L + C_{16} L)d_2 \cos \alpha_2 S C B = 0 \]
For $q_{11} = z_3$

\[ m_3 \ddot{z}_3 = -K_7(SAC - eSAD - SAI) - K_8(SAC + eSAD - SAI) - K_{11} \delta_3(SAC - g_1 SAD - SAI) - g_1 SAD - SAI + GAP) - K_{12} \delta_4(SAC + g_1 SAD + SAI + GAP) + K_{17}(SBA - h_3 SBB + SBC) + K_{18}(SBA + h_3 SBB + SBC) + K_{19}(SBA - h_3 SBB - SBC) + K_{20}(SBA + h_3 SBB - SBC) + m_3 g + KBOM \delta_{11}(SBA - h_3 SBB + TL) + KBOM \delta_{12}(SBA + h_3 SBB + TL) + C_{17}(SBD - h_3 SBE + SBF) + C_{18}(SBD + h_3 SBE + SBF) + C_{20}(SBD + h_3 SBE - SBF) = 0

For $q_{12} = \psi_3$

\[ (I_x^3 \sin^2 \alpha_3 + I_y^3) \ddot{\psi}_3 + (I_x^3 \sin \alpha_3 - I_y^3 \sin \alpha_3) \ddot{\phi}_3 + I_x^3 \cos \alpha_3 \dot{\phi}_3 \cos \psi_3 \cos \alpha_3 + I_y^3 \sin \alpha_3 \dot{\phi}_3 \sin \psi_3 \dot{\phi}_3 \cos \psi_3 + I_y^3 \sin \alpha_3 \dot{\psi}_3 \cos \alpha_3 - I_y^3 \sin \alpha_3 \dot{\psi}_3 \cos \alpha_3 \sin \alpha_3 + I_x^3 \cos \alpha_3 \dot{\psi}_3 \cos \alpha_3 - I_x^3 \cos \alpha_3 \sin \alpha_3 \dot{\psi}_3 + I_y^3 \sin \alpha_3 \psi_3 \dot{\psi}_3 + I_y^3 \cos \alpha_3 \cos \alpha_3 \sin \alpha_3 + I_y^3 \sin \alpha_3 \cos \alpha_3 \sin \alpha_3 + I_x^3 \cos \alpha_3 \dot{\phi}_3 \sin \psi_3 + I_x^3 \cos \alpha_3 \dot{\phi}_3 \sin \psi_3 \sin \alpha_3 + I_y^3 \sin \alpha_3 \dot{\phi}_3 \sin \psi_3 \sin \alpha_3 + I_y^3 \dot{\phi}_3 \sin \psi_3 \sin \alpha_3 + I_y^3 \dot{\phi}_3 \sin \psi_3 \sin \alpha_3 + K_{7e} \cos \psi_3 (SAC - eSAD - SAI) + K_{8e} \cos \psi_3 (SAC + eSAD - SAI) + K_{11} \delta_3 g_1 \cos \psi_3 (SAC - g_1 SAD - SAI + GAP) - K_{12} \delta_4 g_1 \cos \psi_3 (SAC + g_1 SAD - SAI + GAP) - K_{17} h_3 \cos \psi_3 (SBA - h_3 SBB + SBC) + K_{18} h_3 \cos \psi_3 (SBA + h_3 SBB + SBC) + K_{19} h_3 \cos \psi_3 (SBA + h_3 SBB - SBC) + K_{20} h_3 \cos \psi_3 (SBA - h_3 SBB - SBC) + (K_{17} L + K_{18} L) r_3 \cos \psi_3 SCG + (K_{19} L + K_{20} L) r_3 \cos \psi_3 SCH - KBOM \delta_{11} h_3 \cos \psi_3 (SBA - h_3 SBB + TL) + KBOM \delta_{12} h_3 \cos \psi_3 (SBA + h_3 SBB + TL) - C_{17} h_3 \cos \psi_3 (SBD - h_3 SBE + SBF) + C_{18} h_3 \cos \psi_3 (SBD + h_3 SBE + SBF) - C_{19} h_3 \cos \psi_3 (SBD - h_3 SBE + SBF) - C_{20} h_3 \cos \psi_3 (SBD + h_3 SBE - SBF) - (C_{17} L + C_{18} L) r_3 \cos \psi_3 SCC - (C_{19} L + C_{20} L) r_3 \cos \psi_3 SCG = 0
For $q_{13} = \phi_3$

$$
\begin{align*}
(I_{x_3} - I_{y_3})\sin \alpha_3 \dot{\psi}_3 + (I_{x_3} + I_{y_3} \sin^2 \alpha_3 + I_{z_3} \sin^2 \psi_3) \ddot{\phi}_3 \\
+ I_{z_3} \sin \psi_3 \ddot{u}_3 - I_{x_3} \cos \alpha_3 \sin \psi_3 \dot{\psi}_3 (\dot{\phi}_3 \cos \psi_3 \cos \alpha_3 + \psi_3 \sin \alpha_3) \\
- I_{x_3} \cos \psi_3 \sin \alpha_3 (\dot{\phi}_3 \cos \psi_3 \cos \alpha_3 + \psi_3 \sin \alpha_3) \\
+ I_{x_3} \cos \psi_3 \cos \alpha_3 (\dot{\phi}_3 \cos \psi_3 \cos \alpha_3 - \dot{\phi}_3 \cos \psi_3 \sin \alpha_3) \\
+ \psi_3 \cos \alpha_3 \ddot{u}_3 + I_{y_3} \sin \alpha_3 \sin \psi_3 \dot{\psi}_3 (\dot{\phi}_3 \cos \psi_3 \cos \alpha_3 - \dot{\phi}_3 \cos \psi_3 \sin \alpha_3) \\
- I_{y_3} \cos \psi_3 \cos \alpha_3 (\dot{\phi}_3 \cos \psi_3 \cos \alpha_3 - \dot{\phi}_3 \cos \psi_3 \sin \alpha_3) \\
- I_{y_3} \cos \psi_3 \sin \alpha_3 (-\dot{\phi}_3 \cos \alpha_3 \sin \phi_3 + \dot{\phi}_3 \sin \alpha_3 \sin \phi_3) \\
- \dot{\phi}_3 \cos \psi_3 \cos \alpha_3 (\psi_3 \sin \alpha_3 - \dot{\phi}_3 \cos \psi_3 \sin \alpha_3) \\
+ I_{z_3} \sin \psi_3 \ddot{\phi}_3 \cos \psi_3 \dot{\psi}_3 + K_{17}d_3 \cos \phi_3 (SBA - h_3 SBB + SBC) \\
+ K_{18}d_3 \cos \phi_3 (SBA + h_3 SBB + SBC) - K_{19}d_3 \cos \phi_3 (SBA + h_3 SBB - SBC) \\
- K_{20}d_3 \cos \phi_3 (SBA + h_3 SBB - SBC) + K_{25} \ddot{\phi}_3 (\phi_3 - \phi_5) \\
+ C_{17}d_3 \cos \phi_3 (SBD - h_3 SBE + SBF) + C_{18}d_3 \cos \phi_3 (SBD + h_3 SBE + SBF) \\
- C_{19}d_3 \cos \phi_3 (SBD - h_3 SBE - SBF) - C_{20}d_3 \cos \phi_3 (SBD + h_3 SBE - SBF) \\
- SBF = 0
\end{align*}
$$

For $q_{14} = \alpha_3$

$$
\begin{align*}
I_{z_3} (\ddot{\phi}_3 \sin \psi_3 + \dot{\phi}_3 \cos \psi_3 \dot{\psi}_3 + \ddot{\alpha}_3) - I_{x_3} (-\dot{\phi}_3 \cos \psi_3 \sin \alpha_3 \\
+ \dot{\psi}_3 \cos \alpha_3) (-\ddot{\phi}_3 \cos \psi_3 \cos \alpha_3 + \dot{\psi}_3 \sin \alpha_3) - I_{y_3} (-\dot{\psi}_3 \sin \alpha_3 \\
- \dot{\phi}_3 \cos \psi_3 \cos \alpha_3) (\dot{\psi}_3 \cos \alpha_3 - \dot{\phi}_3 \cos \psi_3 \sin \alpha_3) - (K_{17}L \\
+ K_{18}L) \ddot{d}_3 \cos \alpha_3 \dot{\phi} \dot{d} \ddot{c} + (K_{19}L + K_{20}L) \ddot{d}_3 \cos \alpha_3 \dot{\phi} \dot{d} \ddot{c} + K_{25} (\alpha_3 - \alpha_5) \\
- (C_{17}L + C_{18}L) \ddot{d}_3 \cos \alpha_3 \dot{\phi} \dot{d} \ddot{c} + (C_{19}L + C_{20}L) \ddot{d}_3 \cos \alpha_3 \dot{\phi} \dot{d} \ddot{c} = 0
\end{align*}
$$

For $q_{15} = x_4$

$$
\begin{align*}
m_4 \ddot{x}_4 + K_{21}L (x_4 - SBO - SBP + R_9) + K_{22}L (x_4 - SBO - SBP - R_{10}) \\
+ K_{23}L (x_4 + SBO - SBP + R_{11}) + K_{24}L (x_4 + SBO - SBP - R_{12}) \\
- (C_{13}L + C_{14}L) \ddot{S} \ddot{C} + (C_{15}L + C_{16}L) \ddot{S} \ddot{C} - KGIB \ddot{S} (SAF + GIB) \\
- KGIB \ddot{S} (SAF + GIB) = 0
\end{align*}
$$
For $g_{1.6} = z_4$

$m_4 \ddot{z}_4 - K_{13}(SAM - h_2 SAO + SAQ) - K_{14}(SAM + h_2 SAO + SAQ) - K_{15}(SAM - h_2 SAO - SAQ) - K_{16}(SAM + h_2 SAO - SAQ) + K_{21}(z_4 - SBM + SBN - V_1) + K_{22}(z_4 + h_4 \sin \psi_4 + SBN - V_2) + K_{23}(z_4 - SBM - SBN - V_3) + K_{24}(z_4 + SBM - SBN - V_4) - KBOM\delta_9(SAM - h_2 SAO + TL) - KBOM\delta_{1.0}(SAM + h_2 SAO + TL) + m_4 g - C_{1.3}(SAN - h_2 SAP + SAR) - C_{1.4}(SAN + h_2 SAP + SAR) - C_{1.5}(SAN - h_2 SAP - SAR) - C_{1.6}(SAN + h_2 SAP - SAR) = 0$

For $g_{1.7} = \psi_4$

$I_{x_4} \cos \alpha_4 \dot{\alpha}_4(\dot{\phi}_4 \cos \psi_4 \cos \alpha_4 + \dot{\psi}_4 \sin \alpha_4)
+ I_{y_4} \sin \alpha_4(\ddot{\phi}_4 \cos \psi_4 \cos \alpha_4 - \dot{\phi}_4 \cos \psi_4 \sin \psi_4
- \dot{\phi}_4 \cos \psi_4 \sin \alpha_4 \dot{\alpha}_4 + \dot{\psi}_4 \sin \alpha_4 + \ddot{\psi}_4 \cos \alpha_4 \dot{\alpha}_4)
+ I_{x_4} \sin \alpha_4 \dot{\alpha}_4(\dot{\psi}_4 \cos \psi_4 \cos \alpha_4 - \dot{\phi}_4 \cos \psi_4 \sin \alpha_4)
+ I_{y_4} \cos \alpha_4(\ddot{\psi}_4 \cos \psi_4 \cos \alpha_4 - \dot{\psi}_4 \sin \alpha_4 \dot{\alpha}_4 - \ddot{\phi}_4 \cos \psi_4 \sin \alpha_4
+ \dot{\phi}_4 \sin \alpha_4 sin \psi_4 \dot{\psi}_4 - \ddot{\phi}_4 \cos \psi_4 \cos \alpha_4 \dot{\alpha}_4) +
+ I_{x_4} \sin \psi_4 \dot{\phi}_4 \cos \alpha_4(\dot{\phi}_4 \cos \psi_4 \cos \alpha_4 + \dot{\psi}_4 \sin \alpha_4)
- I_{x_4} \sin \psi_4 \dot{\phi}_4 \sin \alpha_4(\dot{\psi}_4 \cos \psi_4 \cos \alpha_4 - \ddot{\phi}_4 \cos \psi_4 \sin \alpha_4)
- I_{z_4} \cos \psi_4(\dot{\phi}_4 \sin \psi_4 + \dot{\alpha}_4) + K_{13} h_2 \cos \psi_4(SAM - h_2 SAO + SAQ)
- K_{14} h_2 \cos \psi_4(SAM + h_2 SAO + SAQ) + K_{15} h_2 \cos \psi_4(SAM - h_2 SAO - SAQ) - (K_{13} L
+ K_{14} L) r_4 \cos \psi_4 SCE - (K_{15} L + K_{16} L) r_4 \cos \psi_4 SCF - K_{21} h_4 \cos \psi_4(z_4 - SBM + SBN - V_1) + K_{22} h_4 \cos \psi_4(z_4 + SBM + SBN - V_2) + K_{23} h_4 \cos \psi_4(z_4 - SBM - SBN - V_3) + K_{24} h_4 \cos \psi_4(z_4 + SBM - SBN - V_4) - K_{21 L R} \cos \psi_4(x_4 - SBO - SBP + R_9) - K_{22 L R} \cos \psi_4(x_4 - SBO - SBP - R_{10}) - K_{23 L R} \cos \psi_4(x_4 + D \sin \alpha_4 - R \sin \psi_4 + R_{11}) - K_{24 L R} \cos \psi_4(x_4 + D \sin \alpha_4 - R \sin \psi_4 - R_{12}) + KBOM\delta_9 h_2 \cos \psi_4(SAM - h_2 SAO + TL) - KBOM\delta_{1.0} h_2 \cos \psi_4(SAM - h_2 SAO - TL) - KBOM\delta_{1.1} h_2 \cos \psi_4(SAM + h_2 SAO + TL) - KBOM\delta_{1.2} h_2 \cos \psi_4(SAM + h_2 SAO - TL) = 0$
\[ \begin{align*}
\text{For } q_{18} &= \phi_4 \\
-I_x^4 \cos \alpha_4 \sin \psi_4 \dot{\psi}_4 (\dot{\phi}_4 \cos \psi_4 \cos \alpha_4 + \dot{\psi}_4 \sin \alpha_4) \\
-I_x^4 \cos \psi_4 \sin \alpha_4 \ddot{\alpha}_4 (\dot{\phi}_4 \cos \psi_4 \cos \alpha_4 + \dot{\psi}_4 \sin \alpha_4) \\
+ I_x^4 \cos \psi_4 \cos \alpha_4 (\dot{\phi}_4 \cos \psi_4 \cos \alpha_4 - \dot{\phi}_4 \cos \alpha_4 \sin \psi_4 \dot{\psi}_4) \\
+ \dot{\phi}_4 \cos \psi_4 \sin \alpha_4 \ddot{\alpha}_4 + \dot{\psi}_4 \sin \alpha_4 + \dot{\psi}_4 \cos \alpha_4 \ddot{\alpha}_4) \\
+ I_y^4 \sin \alpha_4 \sin \psi_4 \dot{\psi}_4 (\dot{\psi}_4 \cos \alpha_4 - \dot{\phi}_4 \cos \alpha_4 \sin \alpha_4) \\
- I_y^4 \cos \psi_4 \cos \alpha_4 \ddot{\alpha}_4 (\dot{\psi}_4 \cos \alpha_4 - \dot{\phi}_4 \cos \alpha_4 \sin \alpha_4) \\
+ I_y^4 \cos \psi_4 \sin \alpha_4 (\dot{\psi}_4 \cos \alpha_4 - \dot{\psi}_4 \sin \alpha_4 \ddot{\alpha}_4 - \dot{\phi}_4 \cos \psi_4 \sin \alpha_4) \\
+ \dot{\phi}_4 \sin \alpha_4 \sin \psi_4 \dot{\psi}_4 - \dot{\phi}_4 \cos \psi_4 \cos \alpha_4 \ddot{\alpha}_4 + I_z^4 \cos \psi_4 \dot{\psi}_4 (\dot{\psi}_4 \sin \psi_4 + \ddot{\psi}_4 \cos \psi_4 \dot{\psi}_4 + \ddot{\alpha}_4) - K_{13d_4} \cos \phi_4 (\text{SAM} - \text{h}_2 \text{SAO} + \text{SAQ}) - K_{14d_4} \cos \phi_4 (\text{SAM} + \text{h}_2 \text{SAO} + \text{SAQ}) + K_{15d_4} \cos \phi_4 (\text{SAM} - \text{h}_2 \text{SAO} - \text{SAQ}) + K_{21D} \cos \phi_4 (z_4 - \text{SBM} + \text{SBN} - V_1) + K_{22D} \cos \phi_4 (z_4 + \text{SBM} + \text{SBN} - V_2) - K_{23D} \cos \phi_4 (z_4 - \text{SBM} + \text{SBN} - V_3) - K_{24D} \cos \phi_4 (z_4 + \text{SBM} - \text{SBN} - V_4) - K_{24D} (\phi_2 - \phi_4) - C_{13d_4} \cos \phi_4 (\text{SAN} - \text{h}_2 \text{SAP} + \text{SAR}) - C_{14d_4} \cos \phi_4 (\text{SAN} + \text{h}_2 \text{SAP} + \text{SAR}) + C_{15d_4} \cos \phi_4 (\text{SAN} - \text{h}_2 \text{SAP} - \text{SAR}) + C_{16d_4} \cos \phi_4 (\text{SAN} + \text{h}_2 \text{SAP} - \text{SAR}) = 0 \\
\text{For } q_{19} &= \alpha_4 \\
I_z^4 (\dot{\phi}_4 \sin \psi_4 + \ddot{\phi}_4 \cos \psi_4 \dot{\psi}_4 + \ddot{\alpha}_4) - I_x^4 (- \ddot{\phi}_4 \cos \psi_4 \sin \alpha_4) \\
+ \dot{\psi}_4 \cos \alpha_4 (\ddot{\phi}_4 \cos \psi_4 \cos \alpha_4 + \dot{\psi}_4 \sin \alpha_4) - I_y^4 (- \ddot{\psi}_4 \sin \alpha_4) \\
- \ddot{\phi}_4 \cos \psi_4 \cos \alpha_4 (\dot{\psi}_4 \cos \alpha_4 - \ddot{\phi}_4 \cos \psi_4 \sin \alpha_4) + (K_{13L} + K_{14L})d_4 \cos \alpha_4 \text{SCE} - (K_{13L} + K_{15L})d_4 \cos \alpha_4 \text{SCF} - K_{21D} \cos \alpha_4 (x_4 - \text{SBO} - \text{SBP} + R_9) - K_{22L} \cos \alpha_4 (x_4 - \text{SBO} - \text{SBP} - R_{10})
\end{align*} \]
\[ + K_{23} LD \cos \alpha_4 (x_4 + SBO - SBP + R_{11}) + K_{24} LD \cos \alpha_4 (x_4 + SBO) \]
\[ - SBP - R_{12} - KT_{24} (\alpha_2 - \alpha_4) + (C_{13} L + C_{14} L) d_4 \cos \alpha_4 SCA \]
\[ - (C_{15} L + C_{16} L) d_4 \cos \alpha_4 SCB = 0 \]

For \( q_{20} = x_5 \)
\[ m_5 \ddot{x}_5 - (K_{17} L + K_{18} L) SCG - (K_{19} L + K_{20} L) SCH + K_{25} L (x_5 - D \sin \alpha_5) \]
\[ - R \sin \psi_5 + R_1 ) + K_{26} L (x_5 - D \sin \alpha_5 - R \sin \psi_5 - R_{14} ) \]
\[ + K_{27} L (x_5 + D \sin \alpha_5 - R \sin \psi_5 + R_{15} ) + K_{28} L (x_5 + D \sin \alpha_5) \]
\[ - R \sin \psi_5 - R_{16} ) - KGIB\dot{\psi}_7 (SAU - GIB) - KGIB\dot{\psi}_9 (SAU + GIB) \]
\[ - (C_{17} L + C_{18} L) SCC - (C_{19} L + C_{20} L) SCD = 0 \]

For \( q_{21} = z_5 \)
\[ m_5 \ddot{z}_5 - K_{17} (SBA - h_3 SBB + SBC) - K_{18} (SBA + h_3 SBB + SBC) - K_{19} (SBA \]
\[ - h_3 SBB - SBC) - K_{20} (SBA + h_3 SBB - SBC) + K_{25} (z_5 - SBQ + SBR - V_5) \]
\[ + K_{26} (z_5 + SBQ + SBR - V_6) + K_{27} (z_5 - SBQ + SBR - V_7) + K_{28} (z_5 \]
\[ + SBQ - SBR - V_8) + m_5 g - KEOM\dot{\phi}_1 (SBA - h_3 SBB + TL) \]
\[ - KEOM\dot{\phi}_2 (SBA + h_3 SBB + TL) - C_{17} (SBD - h_3 SBE + SBF) - C_{18} (SBD \]
\[ + h_3 SBE + SBF) - C_{19} (SBD - h_3 SBE - SBF) - C_{20} (SBD + h_3 SBE - SBF) = 0 \]

For \( q_{22} = \psi_5 \)
\[ (I_{x_5} \sin^2 \alpha_5 + I_{y_5} \cos^2 \alpha_5) \ddot{\psi}_5 + (I_{x_5} \sin \alpha_5 \cos \psi_5 \cos \alpha_5) \]
\[ - I_{y_5} \cos \alpha_5 \cos \psi_5 \sin \alpha_5 \dot{\phi}_5 + I_{x_5} \cos \alpha_5 \ddot{\alpha}_5 (\dot{\phi}_5 \cos \psi_5 \cos \alpha_5) \]
\[ + \dot{\psi}_5 \sin \alpha_5) + I_{x_5} \sin \alpha_5 (- \dot{\phi}_5 \cos \alpha_5 \sin \psi_5 \ddot{\psi}_5 - \dot{\phi}_5 \cos \psi_5 \sin \alpha_5 \dot{\alpha}_5 \]
\[ + \dot{\psi}_5 \cos \alpha_5 \dot{\alpha}_5) - I_{y_5} \sin \alpha_5 \dot{\alpha}_5 (\dot{\psi}_5 \cos \alpha_5 - \dot{\phi}_5 \cos \psi_5 \sin \alpha_5 \]
\[ + I_{y_5} \cos \alpha_5 (- \ddot{\psi}_5 \sin \alpha_5 \dot{\alpha}_5 + \ddot{\phi}_5 \sin \alpha_5 \sin \psi_5 \ddot{\psi}_5 - \ddot{\phi}_5 \cos \psi_5 \cos \alpha_5 \dot{\alpha}_5) \]
\[ + I_{x_5} \dot{\phi}_5 \cos \alpha_5 \sin \psi_5 (\ddot{\phi}_5 \cos \psi_5 \cos \alpha_5 + \ddot{\psi}_5 \sin \alpha_5) \]
\[ - I_{y_5} \ddot{\psi}_5 \sin \alpha_5 \sin \psi_5 (\dot{\phi}_5 \cos \psi_5 \alpha_5 - \dot{\phi}_5 \cos \psi_5 \sin \alpha_5) \]
\[ - I_{x_5} \ddot{\phi}_5 \cos \psi_5 (\ddot{\phi}_5 \sin \psi_5 + \dot{\alpha}_5) + K_{17} h_3 \cos \psi_5 (SBA - h_3 SBB + SBC) \]
\[ - K_{18} h_3 \cos \psi_5 (SBA + h_3 SBB + SBC) + K_{19} h_3 \cos \psi_5 (SBA - h_3 SBB \]
\[ - SBC) - K_{20} h_3 \cos \psi_5 (SBA + h_3 SBB - SBC) - (K_{17} L + K_{18} L) r_5 \cos \psi_5 SCG \]
\[ -(K_{19L} + K_{20L})x_5 \cos \psi_5 SCH - K_{25h_5} \cos \psi_5 (z_5 - SBQ + SBR - V_5) + K_{28h_5} \cos \psi_5 (z_5 + SBQ + SBR - V_5) - K_{27h_5} \cos \psi_5 (z_5 - SBQ - SBR - V_7) + K_{28h_5} \cos \psi_5 (z_5 + SBQ - SBR - V_6) - K_{25LR} \cos \psi_5 (x_5 - SBS - SBT + R_{13}) - K_{28LR} \cos \psi_5 (x_5 - SBS - SBT - R_{14}) - K_{27LR} \cos \psi_5 (x_5 + SBS - SBT + R_{15}) - K_{28LR} \cos \psi_5 (x_5 + SBS - SBT - R_{16}) + KBOM_1 h_3 (SBA - h_3 SBB + TL) - KBOM_3 h_3 (z_5 - z_5 + h_3 SBB + TL) + C_{17h_3} \cos \psi_5 (SBD - h_3 SBE - SBF) - C_{18h_3} \cos \psi_5 (SBD + h_3 SBE + SBF) - C_{19h_3} \cos \psi_5 (SBD - h_3 SBE - SBF) - C_{20h_3} \cos \psi_5 (SBD + h_3 SBE - SBF) - (C_{17L} + C_{18L}) r_5 \cos \psi_5 SCC - (C_{19L} + C_{20L}) r_5 \cos \psi_5 SCD = 0 \]

For \( q_{23} = \phi_5 \)

\[
(I_{x_5} - I_{y_5}) \sin \alpha_5 \dot{y}_5 + (I_{x_5} + I_{y_5}) \sin^2 \alpha_5 + I_{z_5} \sin^2 \psi_5) \ddot{y}_5 + I_{z_5} \sin \psi \ddot{a}_5 - I_{x_5} \cos \alpha_5 \sin \psi \dot{y}_5 \ddot{a}_5 (\dot{\phi}_5 \cos \psi_5 \cos \alpha_5 + \dot{\psi}_5 \sin \alpha_5) - I_{x_5} \cos \psi_5 \sin \alpha_5 \dot{a}_5 (\dot{\phi}_5 \cos \psi_5 \cos \alpha_5 + \dot{\psi}_5 \sin \alpha_5) + I_{x_5} \cos \psi_5 \cos \alpha_5 (\dot{a}_5 \ddot{a}_5 \dot{\phi}_5 + \dot{\phi}_5 \cos \psi_5 \sin \alpha_5) + \dot{\psi}_5 \cos \alpha_5 \dot{a}_5) + I_{y_5} \sin \alpha_5 \sin \psi_5 \ddot{y}_5 \psi_5 \cos \alpha_5 + \dot{\phi}_5 \cos \psi_5 \sin \alpha_5) - I_{y_5} \cos \psi_5 \cos \alpha_5 \dot{a}_5 (\psi_5 \cos \alpha_5 + \dot{\phi}_5 \cos \psi_5 \sin \alpha_5) - I_{y_5} \cos \psi_5 \sin \alpha_5 (\dot{\psi}_5 \sin \alpha_5 \ddot{a}_5 + \dot{\phi}_5 \sin \alpha_5 \sin \psi_5 \psi_5) - \dot{\phi}_5 \cos \psi_5 \cos \alpha_5 \dot{a}_5) + I_{z_5} \cos \psi_5 \dot{y}_5 \ddot{a}_5 (\dot{\phi}_5 \sin \psi \dot{y}_5 + \dot{\phi}_5 \sin \alpha_5) + I_{z_5} \sin \psi \ddot{y}_5 \ddot{y}_5 - K_{17d_5} \cos \phi_5 (SBA - h_3 SBB + SBC) - K_{18d_5} \cos \phi_5 (SBA + h_3 SBB + SBC) + K_{19d_5} \cos \phi_5 (SBA - h_3 SBB + SBC) + K_{20d_5} \cos \phi_5 (SBA + h_3 SBB + SBC) + K_{25D} \cos \phi_5 (z_5 - SBQ + SBR - V_5 + SBR - V_6) - K_{27D} \cos \phi_5 (z_5 - SBQ - SBR - V_7) - K_{28D} \cos \phi_5 (z_5 + SBQ - SBR - V_6) - K_{27D} \cos \phi_5 (z_5 + SBQ + SBR - V_7) - K_{28D} \cos \phi_5 (z_5 + SBQ + SBR - V_6) - K_{27D} \cos \phi_5 (z_5 + SBQ - SBR - V_7) - \phi_5 - C_{17d_5} \cos \phi_5 (SBD - h_3 SBE + SBF) - C_{18d_5} \cos \phi_5 (SBD - h_3 SBE + SBF) + h_3 SBE + SBF) + C_{19d_5} \cos \phi_5 (SBD + h_3 SBE - SBF) + C_{20d_5} \cos \phi_5 (SBD + h_3 SBE - SBF) = 0
For \( q_{24} = \alpha_5 \)
\[
I_x (\dot{\phi}_5 \sin \psi_5 + \dot{\psi}_5 \cos \psi_5 \ddot{\psi}_5 + \dddot{\psi}_5) - I_{x_5} (- \dot{\phi}_5 \cos \psi_5 \sin \alpha_5 \\
+ \dot{\psi}_5 \cos \alpha_5 (\dot{\phi}_5 \cos \psi_5 \cos \alpha_5 + \dot{\psi}_5 \sin \alpha_5) - I_{y_5} (- \dot{\psi}_5 \sin \alpha_5 \\
- \dot{\phi}_5 \cos \psi_5 \cos \alpha_5 (\dot{\psi}_5 \cos \alpha_5 - \dot{\phi}_5 \cos \psi_5 \sin \alpha_5)
\]
- \( K_{25} L \cos \alpha_5 (x_5 - SBS - SBT + R_{13}) - K_{26} L \cos \alpha_5 (x_5 - SBS \\
- SBT - R_{14}) + K_{27} L \cos \alpha_5 (x_5 + SBS - SBT + R_{15}) \\
+ K_{28} L \cos \alpha_5 (x_5 + D \sin \alpha_5 - R \sin \psi_5 - R_{16}) + (K_{17} L \\
+ K_{18} L) d_5 \cos \alpha_5 SCG - (K_{19} L + K_{20} L) d_5 \cos \alpha_5 SC \cos \alpha_5 SCH - K_{T3 5} (\alpha_3 - \alpha_5) \\
+ (C_{17} L + C_{18} L) d_5 \cos \alpha_5 SCC - (C_{19} L + C_{20} L) d_5 \cos \alpha_5 SCD = 0
\]

For \( q_{25} = x_6 \)
\[
m_6 \ddot{x}_6 + KF(x_6 - x_1) + CF(\dot{x}_6 - \dot{x}_1) = 0
\]

For \( q_{26} = y_6 \)
\[
m_6 \ddot{y}_6 + KF(y_6 - y_1) + CF(\dot{y}_6 - \dot{y}_1) = 0
\]

For \( q_{27} = z_6 \)
\[
m_6 \ddot{z}_6 + KF(z_6 - z_1) + CF(\dot{z}_6 - \dot{z}_1) = 0
\]

The above 27 equations describe the motion of the system we have modelled. Options like centerplate extension pads, lateral springs at the centerplates can easily be introduced to the model and the corresponding potential and dissipation energy terms can be derived by the same method.
APPENDIX B

COMPUTER PROGRAM LISTINGS
MAIN PROGRAM

THE FOLLOWING ARE SYSTEM PARAMETERS

SPRING STIFFNESS (LB/IN)

CENTER PLATE SPRING STIFFNESS
SIDE BEARINGS OF FRONT AND REAR BOLSTERS
VERTICAL SUSPENSION SPRING STIFFNESS
LATERAL SPRING CONSTANTS AT POINTS 13, 14, 15, 16
LATERAL SPRING Constants at points 17, 18, 19, 20.
TRACK VERTICAL SPRINGS
TRACK LATERAL SPRINGS
GIANT SPRING
ADUMI BOTTOM SPRING

EQUIVALENT VISCOUS DAMPLING COEFFICIENTS FOR THE SYSTEM: (LB-SEC/IN)

VISCOS DAMPLING COEFFICIENTS FOR ELEMENTS AT POINTS
BETWEEN FRONT HOLSTER AND SIDES 6, 7, 8, 9
BETWEEN FRONT HOLSTER AND TRUCK
BETWEEN REAR BOLSTER AND TRUCK
BETWEEN REAR BOLSTER AND TRUCK

MOMENT OF INERTIA OF THE BODIES IN THE SYSTEM: (IN-LB-SEC**2)

MOMENT OF INERTIA OF THE CAR BODY WITH RESPECT TO ITS CENTER OF MASS ABOUT ITS LATERAL AXIS, LONGITUDINAL AXIS, AND VERTICAL AXIS RESPECTIVELY
MOMENT OF INERTIA OF FRONT BOLSTER WITH RESPECT TO ITS CENTER OF MASS ABOUT ITS LATERAL, LONGITUDINAL, AND VERTICAL AXES
MOMENT OF INERTIA OF REAR BOLSTER WITH RESPECT TO ITS CENTER OF MASS ABOUT ITS LATERAL, LONGITUDINAL, AND VERTICAL AXES RESPECTIVELY
MOMENT OF INERTIA OF FRONT TRUCK WITH RESPECT TO ITS CENTER OF MASS ABOUT ITS LATERAL, LONGITUDINAL, AND VERTICAL AXES RESPECTIVELY
MOMENT OF INERTIA OF REAR TRUCK WITH RESPECT TO ITS CENTER OF MASS ABOUT ITS LATERAL, LONGITUDINAL, AND VERTICAL AXES RESPECTIVELY

MASS OF THE BODIES OF THE SYSTEM: (LB-SEC**2/IN)

MASS OF THE CAR BODY
MASS OF THE FRONT BOLSTER
MASS OF THE REAR BOLSTER
<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>D11</td>
<td>ROLL</td>
<td>-</td>
</tr>
<tr>
<td>D12</td>
<td>YAW</td>
<td>-</td>
</tr>
<tr>
<td>D13</td>
<td>LATERAL DISPLACEMENT OF REAR BOLSTER</td>
<td>-</td>
</tr>
<tr>
<td>D14</td>
<td>LONGITUDINAL</td>
<td>-</td>
</tr>
<tr>
<td>D15</td>
<td>LATERAL DISPLACEMENT OF FRONT TRUCK</td>
<td>-</td>
</tr>
<tr>
<td>D16</td>
<td>HOLL OF REAR BOLSTER</td>
<td>-</td>
</tr>
<tr>
<td>D17</td>
<td>PITCH</td>
<td>-</td>
</tr>
<tr>
<td>D18</td>
<td>YAW</td>
<td>-</td>
</tr>
<tr>
<td>D19</td>
<td>LATERAL DISPLACEMENT OF REAR BOLSTER</td>
<td>-</td>
</tr>
<tr>
<td>D20</td>
<td>LONGITUDINAL</td>
<td>-</td>
</tr>
<tr>
<td>D21</td>
<td>VERTICAL</td>
<td>-</td>
</tr>
<tr>
<td>D22</td>
<td>HOLL OF THE FRONT TRUCK</td>
<td>-</td>
</tr>
<tr>
<td>D23</td>
<td>PITCH</td>
<td>-</td>
</tr>
<tr>
<td>D24</td>
<td>YAW</td>
<td>-</td>
</tr>
<tr>
<td>D25</td>
<td>HOLL OF THE REAR TRUCK</td>
<td>-</td>
</tr>
<tr>
<td>D26</td>
<td>PITCH</td>
<td>-</td>
</tr>
<tr>
<td>D27</td>
<td>YAW</td>
<td>-</td>
</tr>
<tr>
<td>D28</td>
<td>HOLL OF THE REAR TRUCK</td>
<td>-</td>
</tr>
<tr>
<td>D29</td>
<td>PITCH</td>
<td>-</td>
</tr>
<tr>
<td>D30</td>
<td>YAW</td>
<td>-</td>
</tr>
<tr>
<td>D31</td>
<td>TO D31: DISPLACEMENTS OF FREIGHT ELEMENT IN X, Y, AND Z DIRECTIONS</td>
<td>-</td>
</tr>
<tr>
<td>D32</td>
<td>THE DISPLACEMENTS OF FRONT TRUCK</td>
<td>-</td>
</tr>
<tr>
<td>D33</td>
<td>ARE THE CORRESPONDING VELOCITY TERMS</td>
<td>-</td>
</tr>
<tr>
<td>D34</td>
<td>AA(1) TO AA(33) ARE CORRESPONDING ACCELERATION TERMS</td>
<td>-</td>
</tr>
<tr>
<td>D35</td>
<td>COMMON/ACCELERATION</td>
<td>-</td>
</tr>
</tbody>
</table>

**Data for 70-ton Boxcar**

**Dimensions:**
- **D16:** CIx1 = 16,5000
- **D17:** CIY1 = 128,000
- **D18:** CIz1 = 164,000
- **D30:** d1 = 26,40
- **D31:** d2 = 12,000
- **D32:** T1x1 = 120,000
- **D33:** T1y1 = 192,00
- **D34:** T1z1 = 120,000
- **D35:** d = 74,4
<table>
<thead>
<tr>
<th>No.</th>
<th>000173</th>
<th>000171</th>
<th>000172</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C=7.2</td>
<td>D=74</td>
<td>E=0.5</td>
</tr>
<tr>
<td>000173</td>
<td>F1=14.25</td>
<td>K1=72.5</td>
<td>K2=4.5</td>
</tr>
<tr>
<td>000174</td>
<td></td>
<td>K3=4.5</td>
<td></td>
</tr>
<tr>
<td>000175</td>
<td></td>
<td>K4=9.5</td>
<td></td>
</tr>
<tr>
<td>000176</td>
<td></td>
<td>K5=4.4</td>
<td></td>
</tr>
<tr>
<td>000177</td>
<td></td>
<td>H2=39.1</td>
<td></td>
</tr>
<tr>
<td>000178</td>
<td></td>
<td>H3=39.1</td>
<td></td>
</tr>
<tr>
<td>000179</td>
<td></td>
<td>H4=39.1</td>
<td></td>
</tr>
<tr>
<td>000180</td>
<td></td>
<td>H5=39.1</td>
<td></td>
</tr>
<tr>
<td>000182</td>
<td></td>
<td>K6=39.1</td>
<td></td>
</tr>
<tr>
<td>000183</td>
<td></td>
<td>G1=25</td>
<td></td>
</tr>
<tr>
<td>000184</td>
<td></td>
<td>U1=23.7</td>
<td></td>
</tr>
<tr>
<td>000185</td>
<td></td>
<td>D2=26</td>
<td></td>
</tr>
<tr>
<td>000186</td>
<td></td>
<td>D3=26</td>
<td></td>
</tr>
<tr>
<td>000187</td>
<td></td>
<td>D4=26</td>
<td></td>
</tr>
<tr>
<td>000188</td>
<td></td>
<td>D5=26</td>
<td></td>
</tr>
<tr>
<td>000189</td>
<td></td>
<td>B1=68.004</td>
<td></td>
</tr>
<tr>
<td>000190</td>
<td></td>
<td>D1=0.37</td>
<td></td>
</tr>
<tr>
<td>000191</td>
<td></td>
<td>GAP=327.327</td>
<td></td>
</tr>
<tr>
<td>000192</td>
<td></td>
<td>K4=46.6</td>
<td></td>
</tr>
<tr>
<td>000193</td>
<td></td>
<td>C=SPRING STIFFNESS</td>
<td></td>
</tr>
<tr>
<td>000194</td>
<td></td>
<td>K1=6666000</td>
<td></td>
</tr>
<tr>
<td>000195</td>
<td></td>
<td>K2=K1</td>
<td></td>
</tr>
<tr>
<td>000196</td>
<td></td>
<td>K7=K1</td>
<td></td>
</tr>
<tr>
<td>000197</td>
<td></td>
<td>K8=K1</td>
<td></td>
</tr>
<tr>
<td>000198</td>
<td></td>
<td>K5=666000</td>
<td></td>
</tr>
<tr>
<td>000199</td>
<td></td>
<td>K6=666000</td>
<td></td>
</tr>
<tr>
<td>000200</td>
<td></td>
<td>K11=6660000</td>
<td></td>
</tr>
<tr>
<td>000201</td>
<td></td>
<td>K12=6660000</td>
<td></td>
</tr>
<tr>
<td>000202</td>
<td></td>
<td>K11=10420</td>
<td></td>
</tr>
<tr>
<td>000203</td>
<td></td>
<td>K14=K13</td>
<td></td>
</tr>
<tr>
<td>000204</td>
<td></td>
<td>K15=K13</td>
<td></td>
</tr>
<tr>
<td>000205</td>
<td></td>
<td>K16=K13</td>
<td></td>
</tr>
<tr>
<td>000206</td>
<td></td>
<td>K17=K13</td>
<td></td>
</tr>
<tr>
<td>000207</td>
<td></td>
<td>K18=K13</td>
<td></td>
</tr>
<tr>
<td>000208</td>
<td></td>
<td>K19=K13</td>
<td></td>
</tr>
<tr>
<td>000209</td>
<td></td>
<td>K20=K13</td>
<td></td>
</tr>
<tr>
<td>000210</td>
<td></td>
<td>K13L=4425</td>
<td></td>
</tr>
<tr>
<td>000211</td>
<td></td>
<td>K14L=4425</td>
<td></td>
</tr>
<tr>
<td>000212</td>
<td></td>
<td>K15L=4425</td>
<td></td>
</tr>
<tr>
<td>000213</td>
<td></td>
<td>K16L=4425</td>
<td></td>
</tr>
<tr>
<td>000214</td>
<td></td>
<td>K17L=4425</td>
<td></td>
</tr>
<tr>
<td>000215</td>
<td></td>
<td>K18L=4425</td>
<td></td>
</tr>
<tr>
<td>000216</td>
<td></td>
<td>K19L=4425</td>
<td></td>
</tr>
<tr>
<td>000217</td>
<td></td>
<td>K20L=4425</td>
<td></td>
</tr>
<tr>
<td>000218</td>
<td></td>
<td>K61L=666000</td>
<td></td>
</tr>
<tr>
<td>000219</td>
<td></td>
<td>K60K=666000</td>
<td></td>
</tr>
<tr>
<td>000220</td>
<td></td>
<td>K21=105000</td>
<td></td>
</tr>
<tr>
<td>000221</td>
<td></td>
<td>K22=K21</td>
<td></td>
</tr>
<tr>
<td>000222</td>
<td></td>
<td>K23=K21</td>
<td></td>
</tr>
<tr>
<td>000223</td>
<td></td>
<td>K24=K21</td>
<td></td>
</tr>
<tr>
<td>000224</td>
<td></td>
<td>K25=K21</td>
<td></td>
</tr>
<tr>
<td>000225</td>
<td></td>
<td>K26=K21</td>
<td></td>
</tr>
<tr>
<td>000226</td>
<td></td>
<td>K27=K21</td>
<td></td>
</tr>
<tr>
<td>Line</td>
<td>Value</td>
<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000227</td>
<td>0.007</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000228</td>
<td>0.007</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000229</td>
<td>K2d=K21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000230</td>
<td>K21L=K21*2/3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000231</td>
<td>K22L=K21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000232</td>
<td>K23L=K21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000233</td>
<td>K24L=K21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000234</td>
<td>K25L=K21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000235</td>
<td>K2dL=K21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000236</td>
<td>KT24=K1L*39**2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000237</td>
<td>KT35=KT24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000238</td>
<td>KP34=60000.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000239</td>
<td>KP35=60000.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000240</td>
<td>RF=200.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000241</td>
<td>KC=0.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000242</td>
<td>C1</td>
<td>DEFINE OTHER PARAMETERS:</td>
<td></td>
</tr>
<tr>
<td>000243</td>
<td>CF=18.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000244</td>
<td>CC=0.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000245</td>
<td>FH=150.386.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000246</td>
<td>CH=20000.386.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000247</td>
<td>BM2=81.150.3(32.2*12,)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000248</td>
<td>TM4=7300.3(32.2*12,)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000249</td>
<td>d1X3=d1X2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000250</td>
<td>d1Y3=d1Y2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000251</td>
<td>d1Z3=d1Z2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000252</td>
<td>TIX5=TIX4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000253</td>
<td>TYb=TY4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000254</td>
<td>TIZ5=TIZ4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000255</td>
<td>uM3=6h2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000256</td>
<td>TMST4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000257</td>
<td>SPE=60.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000258</td>
<td>T=0.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000259</td>
<td>DT=0.00025</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000260</td>
<td>TMAX=3.0003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000261</td>
<td>PT=0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000262</td>
<td>PI=ACOS(-1,)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000263</td>
<td>GMAG=PI<em>SPEED</em>17.6/RL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000264</td>
<td>PHI1=PI/2.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000265</td>
<td>PHI2=PI*61/RL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000266</td>
<td>PHI3=PI*U/RL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000267</td>
<td>PHI4=PIH1+PHI3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000268</td>
<td>PHI5=PHI2+PHI3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000269</td>
<td>TL=3.0875</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000270</td>
<td>b=32.2*12,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000271</td>
<td>GAP=6GAP</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000272</td>
<td>GAPDEGAP</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000273</td>
<td>SI=2.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000274</td>
<td>N=66</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000275</td>
<td>L=1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000276</td>
<td>G0=386.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000277</td>
<td>DF=57.29528</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000278</td>
<td>WRITE (6,71) SPEED</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000279</td>
<td>WRITE (6,72) DT</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000280</td>
<td>WRITE (6,70) TMAX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000281</td>
<td>WRITE (6,73) PT</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000282</td>
<td>WRITE (6,75) K1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>000283</td>
<td>WRITE (6,77) K5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(Note: The text appears to be a collection of FORTRAN statements, which likely represent a computer program for simulating the motion of a car body. The program appears to involve solving 30 equations of motion and calculating various parameters such as car body displacement and acceleration. The code includes initialization of variables, assignment of displacement and velocities, and calls to a subroutine named ACCELN for calculating accelerations.)
CALL RUNGE FOR NUMERICAL INTEGRATION.

IF (JT) 9.10.9

DO 30 2 = 1,33

Y(1) = Y(1) + DT*F(1)

DO 30 2 = 1,33

GO TO 212

DO 303 1 = 34,66

AC(I) = AA(I) / GF

DO 303 1 = 34,66

DO 240 1 = 4,6

AC(I) = AA(I) * DF

DO 240 1 = 4,6

DO 370 1 = 1,33

RETURN

HERE Y(I) TO DD(I) & VY(I) FOR THE NEXT TIME STEP.
00390 000  DD(I)=Y(I)
00399 000  SBl VV(I)=Y(I+33)
00400 000  GO TO 842

00401 000  17 STOP
00402 000  END

00403 000  C

00404 000  C  SUBROUTINE DELGAP

00405 000  C

00406 000  C

00407 000  C ******************************************************************************

00408 000  C SUBROUTINE DELGAP CHECKS THE SIDE BEARING CLEARANCE AT EACH TIME

00409 000  C STEP IF THE CLEARANCE IS EQUAL TO ZERO THEN THE SIDE BEARING

00410 000  C COMES INTO THE ACTION

00411 000  C******************************************************************************

00412 000  C SUBROUTINE DELGAP (GAPC,GAPD,SAA,SAB,SAC,SAD,G1,DEL1,DEL2,SAI,

00413 000  C 1UEL3,1UEL4,PTIME,GAP,K5,K6,K11,K12)

00414 000  C

00415 000  C

00416 000  C  REAL K5,K6,K11,K12

00417 000  C

00418 000  C GAP1=(SAA+SAI-G1*SAB)

00419 000  C IF(GAPC+GAP1<2:2

00420 000  C      1 DEL1=1:0

00421 000  C

00422 000  C GO TO 4

00423 000  C

00424 000  C GAP2=(SAA+SAI+G1*SAB)

00425 000  C IF(GAPD+GAP2<=4:4

00426 000  C      3 DEL2=1:0

00427 000  C

00428 000  C GO TO 12

00429 000  C

00430 000  C GAP3=(SAC-SAI-G1*SAD)

00431 000  C IF(GAPC+GAP3<6:6

00432 000  C      5 DEL3=1:0

00433 000  C

00434 000  C GO TO 8

00435 000  C

00436 000  C GAP4=(SAC-SAI+G1*SAD)

00437 000  C IF(GAPD+GAP4<8:8

00438 000  C      7 DEL4=1:0

00439 000  C

00440 000  C GO TO 14

00441 000  C

00442 000  C IF (TIME) 53,54,55

00443 000  C

00444 000  C SF1=K5*DEL1*(SAA-G1*SAB+GAP+SAA)

00445 000  C SF2=K6*DEL2*(SAA-G1*SAB+GAP+SAA)

00446 000  C SF3=K11*DEL3*(SAC-G1*SAD+GAP-SAA)

00447 000  C SF4=K12*DEL4*(SAC+G1*SAD+GAP+SAA)

00448 000  C RXN=-(SF1+SF2+SF3+SF4) /1000

00449 000  C WRITE (6d6) RXN

00450 000  C

00451 000  C RETURN

00452 000  C

00453 000  C SUBROUTINE RUNG

00454 000  C

UTILITY NETWORK OF AMERICA
THIS SUBROUTINE INTEGRATES EQUATIONS OF MOTION BY RUNGE-KUTTA

SUBROUTINE RUNG(N,F,T,Y,DT,JT)

DIMENSION SAVEY(66),PHI(66),Y(66),F(66)

SUBROUTINE RUNG(N,F,T,Y,DT,JT)

RETURN

SUBROUTINE SPRING CHECKS FOR THE BOTTOMING OF SUSPENSION SPRINGS.
IF THE SPRINGS BOTTOM OUT, A SPRING OF MUCH GREATER STIFFNESS IS
ADDED IN PARALLEL TO THE SPRING GROUPS.

SUBROUTINE SPRING (SAM,SAO,SBA,SBB,H2,H3,TL,TIME,DEL9,DEL10,
1DELI1,DELI2)

IF (SAM>H2*SAO+TL) 1r2*2
1 DEL9=1.
GO TO 11

2 DEL9=0.

11 IF (SAM+H2*SAO+TL) 3,4,4
3 DEL10=1.
GO TO 12

4 DEL10=0.

12 IF (SBA>H3*SBB+TL) 5,6,6
5 DEL11=1.
GO TO 12

6 DEL11=0.
SUBROUTINE DELGIB checks the GIB (LAT.) CLEARANCE between THE
BOLSTER AND SIDEFRAME. EACH TIME IT IS CALLED, IF GIB IS ZERO
THE GIB SPRINGS ARE ADDED IN PARALLEL TO THE LAT. SPRING GROUPS.

SUBROUTINE DELGIB(SAF,SAU,GIB,DEL5,DEL6,DEL7,DEL8,T,PTIME)

IF (SAF=GIB) 1:1:2

DEL5=1.
GO TO 4

IF (SAF+GIB) 3:4:4

DEL6=1.
GO TO 9

DEL6=0.

IF (SAU=GIB) 5:6:6

DEL7=1.

DEL7=0.

IF (SAU+GIB) 7:8:8

DEL8=1.
GO TO 14

DEL8=0.

IF (T-PTIME) 5:6:54

WRITE (6,16) DEL5,DEL6,DEL7,DEL8

F0RMAT (/10X,*DEL5=*F4.1,*DEL6=*F4.10X,*DEL7=*F4.110X,*DEL8=*F4.110X),

1' *DEL8=*F4.1)

RETURN

END

SUBROUTINE SGNFUN

SUBROUTINE SGNFUN computes the RELATIVE VEL. BETWEEN THE
BOLSTER AND THE SIDEFRAME COLUMN AND Assigns THE PROPER SIGN
TO THE DAMPING FORCE SUCH THAT THE DAMPING FORCE ALWAYS
OPPOSES THE MOTION.
SUBROUTINE SGNFUN (SCA,SCB,SCC,SCD,SAN3,SAN4,SAN5,SAN6,
1$BD7,SB$BD9,SB$BD)

IF (SCA) 1,2,3
1 $CA$=1.
GO TO 2

$CA$=1.

2 IF (SCB) 4,5,6
4 $CB$=1.
GO TO 5

6 $CB$=1.

3 IF (SCC) 7,8,9
7 $CC$=1.

6 IF (SCD) 10,11,12
10 $CD$=1.
GO TO 11

12 $CD$=1.

11 IF (SAN3) 13,14,15
13 $AN3$=1.
GO TO 14

15 $AN3$=1.

14 IF (SAN4) 16,17,18
16 $AN4$=1.
GO TO 17

18 $AN4$=1.

17 IF (SAN5) 19,20,21
19 $AN5$=1.
GO TO 20

21 $AN5$=1.

20 IF (SAN6) 22,23,24
22 $AN6$=1.
GO TO 23

24 $AN6$=1.

23 IF (SAN6) 25,26,27
25 $BDT$=1.

26 $BDT$=1.

22 IF (SAN6) 27,28,29
27 $BD$=1.

28 $BD$=1.

21 IF (SAN6) 29,30,31
30 $BD$=1.

31 $BD$=1.

24 IF (SAN6) 32,33,34
32 $BD$=1.
GO TO 35

34 $BD$=1.

33 $BD$=1.

32 IF (SAN6) 35,36,37
35 RETURN

36 RETURN

END
SUBROUTINE CPLATE

************** SUBROUTINE CPLATE COMPUTES THE VERT. REACTIONS OF THE FRONT AND REAR CENTER PLATES **************

SUBROUTINE CPLATE (SAA, SAB, SAI, SAC, SAD, E, K1, K2, K7, K8)

HEAL K1, K2, K7, K8

SP12 = K1 * (SAA - E + SAB + SAI) + K23 * (SAA + E + SAB - SAI)

SP76 = K7 * (SAC - E + SAD - SAI) + K8 * (SAC + E + SAD - SAI)

MXL1 = SP12 / 1000,

MXRNK = SP76 / 1000,

IF (MXL1) 1 1 + 2

IF (MXRNK) 3 3 + 4

WRITE (6,16) RXNFR, RXRNK

16 FORMAT (/10X, FRONT CENTER PLATE VERT. RXN=", F7.1, /10X,

"REAR CENTER PLATE VERT. RXN=", F7.1)

RETURN

END

SUBROUTINE CAL

************** SUBROUTINE CAL COMPUTES THE FRONT BOLSTER LAT. RXN AND WHEEL LOADS **************

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)

SUBROUTINE CAL (K1L, K1R, K1L, K16L, K61B, K21, K23, K26, K28,

1SC, SCF, SAF, SHM, GEN, SHE, SBR, SIR, DEL5, DEL6, V1, V3, V6, V8, DD)
**SUBROUTINE T5000**

**COMPUTES THE VERT. ACCELN AND THE LAT. ACCELN**

**ON THE ROOF AND FLOOR OF THE CAR BODY AT BOTH ENDS.**

**DIMENSION AA(33)**

```
GF=380.4
XM=61.
YN=70.
XL=331.2
```

**VAAN=(AA(3)+XL*AA(5))/GF**

**VABN=(AA(3)-XL*AA(5))/GF**

**A4AN=(AA(1)+XM*AA(4)-XL*AA(6))/GF**

**A4BN=(AA(1)+XM*AA(4)+XL*AA(6))/GF**

**AFAN=(AA(1)-YN*AA(4)-XL*AA(6))/GF**

**AFBN=(AA(1)-YN*AA(4)+XL*AA(6))/GF**

**WRITE (6,16) VAAN, VABN**

**WRITE (6,17) A4AN, A4BN**

**WRITE (6,18) AFAN, AFBN**

**RETURN**

**SUBROUTINE ACCELN**

**THIS SUBROUTINE COMPUTES THE ACCELERATION VALUES**

**SUBROUTINE ACCELN(OD, VV, AA, N)**
<table>
<thead>
<tr>
<th>Line</th>
<th>Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>000797</td>
<td>000 R12=0.</td>
</tr>
<tr>
<td>000799</td>
<td>000 R13=0.</td>
</tr>
<tr>
<td>000799</td>
<td>000 R14=0.</td>
</tr>
<tr>
<td>000800</td>
<td>000 R15=0.</td>
</tr>
<tr>
<td>000800</td>
<td>000 R16=0.</td>
</tr>
<tr>
<td>000800</td>
<td>000 2 A=0.</td>
</tr>
<tr>
<td>000803</td>
<td>000 YS=A<em>SIN(OMGA</em>T)</td>
</tr>
<tr>
<td>000804</td>
<td>000 DYS=A<em>OMGA</em>COS(OMGA*T)</td>
</tr>
<tr>
<td>000807</td>
<td>000 H=16.6</td>
</tr>
<tr>
<td>000807</td>
<td>000 D=34.</td>
</tr>
<tr>
<td>000809</td>
<td>000 XM=48.</td>
</tr>
<tr>
<td>000809</td>
<td>000 XL=500.</td>
</tr>
<tr>
<td>000810</td>
<td>000 Cl5=261.</td>
</tr>
<tr>
<td>000810</td>
<td>000 Cl4=C13</td>
</tr>
<tr>
<td>000812</td>
<td>000 Cl6=C13</td>
</tr>
<tr>
<td>000813</td>
<td>000 Cl7=C13</td>
</tr>
<tr>
<td>000814</td>
<td>000 Cl8=C13</td>
</tr>
<tr>
<td>000816</td>
<td>000 Cl9=C13</td>
</tr>
<tr>
<td>000817</td>
<td>000 C20=C13</td>
</tr>
<tr>
<td>000818</td>
<td>000 C3=0.</td>
</tr>
<tr>
<td>000819</td>
<td>000 C4=0.</td>
</tr>
<tr>
<td>000820</td>
<td>000 C5=0.</td>
</tr>
<tr>
<td>000821</td>
<td>000 C1L=C1C</td>
</tr>
<tr>
<td>000822</td>
<td>000 C14L=C13L</td>
</tr>
<tr>
<td>000823</td>
<td>000 C15L=C13L</td>
</tr>
<tr>
<td>000825</td>
<td>000 C17L=C13L</td>
</tr>
<tr>
<td>000827</td>
<td>000 C18L=C13L</td>
</tr>
<tr>
<td>000828</td>
<td>000 C20L=C13L</td>
</tr>
<tr>
<td>000829</td>
<td>000 C21L=C13L</td>
</tr>
<tr>
<td>000830</td>
<td>000 C GROUP VARIABLES INTO SYMBOLS</td>
</tr>
<tr>
<td>000831</td>
<td>000 C</td>
</tr>
<tr>
<td>000832</td>
<td>000 C</td>
</tr>
<tr>
<td>000833</td>
<td>000 SAnE=DU(3)-DU(9)</td>
</tr>
<tr>
<td>000834</td>
<td>000 SAnE=DU(3)-DU(10)</td>
</tr>
<tr>
<td>000835</td>
<td>000 SAnE=DU(3)-DU(15)</td>
</tr>
<tr>
<td>000836</td>
<td>000 SAnE=DU(4)-DU(10)</td>
</tr>
<tr>
<td>000837</td>
<td>000 SAnE=DU(4)-DU(15)</td>
</tr>
<tr>
<td>000837</td>
<td>000 SAnE=DU(12)-DU(24)</td>
</tr>
<tr>
<td>000838</td>
<td>000 SAnE=DU(14)-DU(24)</td>
</tr>
<tr>
<td>000839</td>
<td>000 SAnE=DU(14)-DU(24)</td>
</tr>
<tr>
<td>000840</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000841</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000842</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000843</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000844</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000845</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000846</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000847</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000848</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000849</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000850</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000851</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000852</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
<tr>
<td>000853</td>
<td>000 SAnE=DU(10)+DU(22)</td>
</tr>
</tbody>
</table>

**UTILITY NETWORK OF AMERICA**
101 CONTINUE

AM1(1,2) = -2 * BIX2 * A1 / (CIX1 + BIM2 + BIM3)

AM1(1,5) = CONS1

AM1(2,1) = -2 * BIX2 * A1 / (CIX1 + BIM2 + BIM3)

AM1(2,3) = CONS4

AM1(2,5) = CONS4

AM1(3,2) = CONS4

AM1(3,4) = CONS4

AM1(3,5) = CONS5

AM1(4,3) = CONS4

AM1(4,5) = CONS6

AM2(1,2) = -2 * BIX2 - R1Y2 * DD(12) / (BIX2 + DD(12) + 2 * BIX2)

AM2(2,3) = CONS5

AM2(2,5) = CONS5

AM2(3,2) = CONS6

AM2(3,4) = CONS6

AM3(1,2) = (BIX3 - R1Y3) * DD(18) / (BIX3 + DD(18) + 2 * R1Y3)

AM3(1,4) = CONS1

AM3(2,1) = (BIX3 - R1Y3) * DD(18) / (BIX3 + DD(18) + 2 * R1Y3)

AM3(2,4) = CONS5

AM3(3,2) = DD(16)

AM3(3,4) = CONS18

AM4(1,2) = (TIX4 - T1Y4) * DD(24) / (TIX4 + DD(24) + 2 * T1Y4)

AM4(1,4) = CONS6

AM4(2,1) = (TIX4 - T1Y4) * DD(24) / (TIX4 + DD(24) + 2 * T1Y4)

AM4(2,3) = T1Z4 * UN(22) / (TIX4 + T1Y4 + DD(24) + 2 * T1Z4 + DD(22) + 2)

AM4(2,4) = CONS23

AM4(3,2) = DU(22)

AM4(3,4) = CONS24

AM5(1,2) = (TIX5 - T1Y5) * DD(30) / (TIX5 + DD(30) + 2 * T1Y5)

AM5(1,4) = CONS24

AM5(2,1) = (TIX5 - T1Y5) * DD(30) / (TIX5 + DD(30) + 2 * T1Y5 + DD(28) + 2)

AM5(2,3) = T1Z5 * UN(28) / (TIX5 + T1Y5 + DD(30) + 2 * T1Z5 + DD(28) + 2)

AM5(3,2) = DU(24)

AM5(3,4) = CONS30

AM6(3,4) = CONS30

CALL LSIMEW TO SOLVE THE MATRIX:

ASSIGN SOLUTION TO CORRESPONDING ACCELN. VARIABLES
001310 000 AA(8) = CONS8
001311 000 AA(9) = CONS9
001312 000 AA(10) = A2(1)
001313 000 AA(11) = A2(2)
001314 000 AA(12) = A2(3)
001315 000 AA(13) = 0,
001316 000 AA(14) = CONS14
001317 000 AA(15) = CONS15
001318 000 AA(16) = A3(1)
001319 000 AA(17) = A3(2)
001320 000 AA(18) = A3(3)
001321 000 AA(19) = CONS19
001322 000 AA(20) = CONS20
001323 000 AA(21) = CONS21
001324 000 AA(22) = A4(1)
001325 000 AA(23) = A4(2)
001326 000 AA(24) = A4(3)
001327 000 AA(25) = CONS25
001328 000 AA(26) = CONS26
001329 000 AA(27) = CONS27
001330 000 AA(28) = A5(1)
001331 000 AA(29) = A5(2)
001332 000 AA(30) = A5(3)
001333 000 AA(31) = CONS31
001334 000 AA(32) = CONS32
001335 000 AA(33) = CONS33
001336 000 IF (T = PTIME) 63, 64, 64
001337 000 C GENERATE INFORMATION FOR COMPARING WITH THE 5000 MILE ROAD TEST DATA
001338 000 C
001339 000 C
001340 000 64 CALL T5000 (AA)
001341 000 C COMPUTE SPRING DEFLECTIONS AT PRINT TIME
001342 000 C
001343 000 C
001344 000 DP45 = S5AM + H2 * SA0
001345 000 DP46 = S5AM + H2 * SA0
001346 000 DP79 = SBA + H3 * SBD
001347 000 DP90 = SBA + H3 * SBD
001348 000 WRITE (6, J4+3) DP35, DP79, DP80
001350 000 1F10.3
001351 000 63 RETURN
001352 000 END

END ELT.
SUBROUTINE DFSR1E(NP,NH,Y,A,B)
DIMENSION Y(NP),A(1),B(1)

C  NUMBER OF HARMONICS MUST NOT EXCEED HALF

C  THE NUMBER OF DATA POINTS INPUTED.

C

NNH=MIN0(NH,NP/2)

C  INITIALIZATION AND CONSTANTS.

SP=0.

CP=1.

RN=2./NP

B(1)=0.

A(1)=0.

ARG=3.14159265

C=COSt(ARG)

S=SinT(ARG)

C

C  COMPUTE A FOR THE ZERO TH HARMONIC

C

U0=N1=1,NP

A(1)=A(1)+Y(I)

A(1)=RN*A(1)

C

C  MAIN LOOP.

C

DO 100 K=1,NNH

X=C*CP-S*SP

SP=C*SP+S*CP

CP=X

U=0.

V=0.

C  COMPUTE RECURSIVE U,S

U0=21F=2 NP.

U=N+1+2.*CP*V-U

U=V

A(K+1)=RN*(Y(I)+CP*V-U)

A(K+1)=RN*SP*V

RETURN

END

LOAD 0285 12/0 0285 -1 X1E05

0-X1E05*MSG: TAPE: 0285 7 TRK RING OUT

0 WJB.

X1E05 FIN WDH006

UTILITY NETWORK OF AMERICA
BEGIN IS FREQ
FON 511A-07/07/15-10:30:40 (+0)

MAIN PROGRAM

STORAGE USED: CODE(I) 00014; DATA(O) 001347; BLANK COMMON(2) 000000

EXTERNAL REFERENCES

0003 DFSKIE
0004 NINIRS
0005 NRDUO
0006 NIOIS
0007 NWDUS
0011 XPRK
0012 NSTOPS

STORAGE ASSIGNMENT

0000 001304 10F 0001 001311 10F 0000 001311 1110 0000 001311 111F 0001 000045 1246 0001 001036 18L
0000 001304 30F 0001 000049 91L 0000 R 000311 A 0000 R 000623 AMP 0000 R 000456 R
0000 R 001307 DELFCY 0000 R 001777 FREQ 0000 R 001306 FREQCY 0000 I 001302 I
0000 I 001303 NH 0000 I 001304 NP 0000 R 001305 PERIOD 0000 R 001313 PSO 0000 R 000000 Y

00101 1* DIMENSION Y(21),A(101),B(101),AMP(101),FREQ(101),PSD(101)
00103 2* 91 READ (5,11),END=18 (Y(I),I=1,201)
00111 3* 111 FORMAT (6X,10F6.3,14X)
00112 4* NH=100
00113 5* NF=201
00114 6* PERIOD=4.5
00115 7* FREQCY=FREQCY
00116 8* DELFCY=FREQCY
00117 9* CALL DFSKIE (NP,NH,Y,A,B)
00120 10* WHITE (6,10)
00122 11* 1U FORMAT (/6X,*FREQUENCY (Hz) **10X,*AMPLITUDE (G) **10X,
00122 12* 1G=2/Hz)!
00123 13* DO 20 I=2,101
00126 14* J=1=1
00127 15* AMP(I)=A(I)**2+B(I)**2**0.5
00130 16* FREQ(I)=DELFCY
00131 17* PSD(I)=AMP(I)**2/DELFCY
00132 18* 2U WHITE (6,30) FREQ(I),AMP(I),PSD(I)
00140 19* 3U FORMAT (/6X,F10.4,13X,F6.3,13X,F6.3)
00141 20* GO TO 91
00142 21* 1E STOP
00143 22* END

END OF COMPILATION: NO DIAGNOSTICS.

UTILITY NETWORK OF AMERICA
APPENDIX C

C-1. EQUIVALENT VISCOS DAMPING COEFFICIENT

C-2. DETERMINATION OF SUSPENSION SPRING STIFFNESS FOR A TYPICAL TRUCK

C-3. BENDING STIFFNESS OF A TYPICAL TRUCK BOLSTER

C-4. DESCRIPTIVE DATA FOR A 70-TON BOX CAR

C-5. TRACK INPUT EQUATIONS
C-1. Equivalent Viscous Damping Coefficient

An equivalent viscous damping coefficient is derived on the principle of equivalent energy dissipation for the coulomb friction losses and losses by linear viscous damping.

For a non-linear damper, let the friction force be $F(x, \dot{x})$ with motion assumed to be $x = A \sin \omega t$. The energy dissipated per cycle in this model then becomes

$$E_{NL} = \int F(x, \dot{x}) \, dx = \int F(x, \dot{x}) \frac{dx}{dt} \, dt = \int_{0}^{\tau} F(x, \dot{x}) \dot{x} \, dt = A\omega \int_{0}^{\tau} F(x, \dot{x}) \cos \omega t \, dt$$

where $\tau = \text{periodic time} = \frac{2\pi}{\omega}$ and

$A = \text{amplitude of the assumed motion.}$

For a linear damper, $F = cx$, where $c$ is the coefficient of the viscous damping system, and the energy dissipated per cycle, for this case is given by

$$E_{L} = \int F \, dx = \int c\dot{x} \, dx = \int_{0}^{\tau} cA\omega \cos \omega t \frac{dx}{dt} \, dt = A^{2}\omega^{2}c \int_{0}^{\tau} \cos^{2} \omega t \, dt = A^{2}\omega^{2}c \frac{\tau}{2} = \pi A^{2}c \omega.$$ 

It is assumed here that both systems, the linear and the nonlinear model, have the same amplitude $A$, and the same periodic time. This model assumption is shown to hold good at a later stage, when the model is compared with test data. Therefore, for $E_{L} = E_{NL}$, we have

$$C_{\text{equiv.}} = \frac{1}{\pi A} \int_{0}^{\tau} F(x, \dot{x}) \cos \omega t \, dt \quad (1)$$

Analysis of a Typical Truck Damping System

Refer to the free body diagram (Fig. C-1) of the friction shoes in the upward stroke, $\sum F_x = 0$ gives

$$N_1 \cos \theta - F_1 \sin \theta - N_2 = 0 \quad (2)$$

$\sum F_y = 0$ gives
FIG. C-1. FREE BODY DIAGRAM OF A TYPICAL FRICTION SHOE DURING UPSTROKE
\[ P - F_2 - N_1 \sin \theta - F_1 \cos \theta = 0 \] (3)

with \( F_1 = N_1 \mu_1 \)

\[ F_2 = N_2 \mu_2 \] (5)

we have four equations with four unknowns, \( F_1, F_2, N_1, N_2 \).

Applying specific values, based on data supplied by a leading truck manufacturer, where \( \mu_1 = 0.5, \mu_2 = 0.37, \theta = 35^\circ \)

\[ P = 3220 \text{ lb} \]

we can solve for \( N_1 \) and \( N_2 \), giving

\[ N_1 = 2730 \text{ lb} \]
\[ N_2 = 1452 \text{ lb} \]

So, \( F_1 \cos \theta = 0.5 \times 2730 \times 0.819 \text{ lb} = 1118 \text{ lb} \)

and \( F_2 = 0.37 \times 1452 \text{ lb} = 537 \text{ lb} \)

Therefore, the total vertical frictional force exerting on the wedge-shaped friction shoe on the upward stroke

\[ F_u = F_1 + F_2 = 1655 \text{ lb} \]

Based on experimental studies performed by a truck manufacturer on a typical truck, it was concluded that the upward damping stroke dissipates 65% of the total energy in the cycle. Therefore, the total damping force per cycle

\[ F = 1655/0.65 \text{ lb} = 2546 \text{ lb} \]

This is the quantity assumed in the earlier analysis (equation (1) for the equivalent viscous damping) to be \( F = f(x,x) \). It is seen that in the case of this typical truck, the force \( F \) can be modeled as a constant in this analysis.

The coefficient of viscous damping then becomes

\[ C = \frac{2546}{T} \int_0^T \cos \omega t \, dt = \frac{10184}{\pi \omega A} \]

This model holds good provided that the basic assumptions relating the linear and nonlinear motions are not violated.

Putting \( \omega = 2\pi f \), assuming \( f \) approximately equal to the frequency of excitation, the amplitude of damping oscillation

\[ A = \frac{1}{2}(\text{suspension group travel length} + \text{static deflection on suspension group}) \]

\[ = \frac{1}{2}(3.6875 + 2.29) \text{ in} \approx 3.0 \text{ in} \]
So, \[ C = \frac{10184}{2\pi^2 f_A} \]
\[ C = 172 \frac{1}{f} \]

This value of C is the equivalent viscous damping coefficient [lb-sec/in] for the typical truck damping system.

C-2. Determination of Suspension Spring Stiffnesses for a Typical Truck

The typical truck considered has three series of suspension springs classified by spring travels. The type with 3 11/16 in. travel is chosen to conform with the suspension type in the test truck of the 5000 Mile Box Car Vibration test. There are two suspension groups/truck and each group has the following stiffnesses:

<table>
<thead>
<tr>
<th>Spring Designation</th>
<th>Spring Rate (lb/in)</th>
<th>No. of Springs</th>
<th>Total Spring Rate (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer D5</td>
<td>2140</td>
<td>7</td>
<td>14980</td>
</tr>
<tr>
<td>Inner D5</td>
<td>1070</td>
<td>3</td>
<td>3210</td>
</tr>
<tr>
<td>Side Spring Outer 4143-2</td>
<td>984</td>
<td>2</td>
<td>1968</td>
</tr>
<tr>
<td>Side Spring Inner 4143-3</td>
<td>439</td>
<td>2</td>
<td>878</td>
</tr>
</tbody>
</table>

Total suspension stiffness/group = 21036

This suspension group stiffness is represented by two springs in the mathematical model, and hence each one has the stiffness of 10518 lb/in. When the truck bolster bending stiffness is incorporated into these springs in the model, their stiffness value will be slightly lower.

C-3. Bending Stiffness of a Typical Truck Bolster

Before computing the bending stiffness of a typical truck bolster, the area moment of inertia of the bolster has to be estimated. Based on the mechanical drawing on the typical truck bolster on p. 831, CAR AND LOCOMOTIVE CYCLOPEDIA, 1966, it is assumed that sections I, II and III are uniform enough to have a constant area moment of inertia of its own (refer to Fig. C-2a).

The area of moment of inertia about its own centroidal axis
is determined for each section and they are respectively

\[ I_{Y-I} = 501.57 \text{ in}^4 \]
\[ I_{Y-II} = 811.3 \text{ in}^4 \]
\[ I_{Y-III} = 1431.3 \text{ in}^4 \]

Assume the truck bolster is simple supported with a concentrated load in the center as shown, the area moments of inertia for each section is plotted in Figure C-2b and the corresponding shear and bending moment diagrams plotted in Figures C-2c and C-2d.

Applying Castigliano's strain energy method, and neglecting strain energy due to torsional and axial loading, the total energy in the truck bolster,

\[ U = \frac{1}{2} \int_0^L \frac{M^2}{EI} \, dx + \frac{K}{2} \int_0^L \frac{Q^2}{AG} \, dx, \]

and so the deflection at \( P \) becomes

\[ \Delta_P = \frac{3U}{3P} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} \, dx + \frac{K}{2} \int_0^L \frac{Q}{AG} \frac{\partial Q}{\partial P} \, dx \]

To find the deflection at the mid-span of the typical truck bolster, the following table is constructed:

<table>
<thead>
<tr>
<th>Load Type</th>
<th>Section I</th>
<th>Section II</th>
<th>Section III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bending</td>
<td>( \frac{P}{2}(a-x) )</td>
<td>( \frac{P}{2}(b-x) )</td>
<td>( \frac{P}{2}(c-x) )</td>
</tr>
<tr>
<td>Transverse Shear</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

Therefore, deflection

\[ \Delta_P = \frac{2}{EI_1} \int_0^a \left( \frac{P}{2} (6.75 - x_1) \right) \frac{1}{2} (6.75 - x_1) \, dx_1 \]
\[ + \frac{2}{EI_2} \int_0^b \left( \frac{P}{2} (28.75 - x_2) \right) \frac{1}{2} (28.75 - x_2) \, dx_2 \]
\[ + \frac{2}{EI_3} \int_0^c \left( \frac{P}{2} (38.75 - x_3) \right) \frac{1}{2} (38.75 - x_3) \, dx_3 \]
\[ + \frac{2K}{A_1G} \int_0^a \left( \frac{P}{2} \right) \left( \frac{1}{2} \right) \, dx_1 + \frac{2K}{A_2G} \int_0^b \left( \frac{P}{2} \right) \left( \frac{1}{2} \right) \, dx_2 \]
\[ + \frac{2K}{A_3G} \int_0^c \left( \frac{P}{2} \right) \left( \frac{1}{2} \right) \, dx_3 \]
FIG. C-2A. CROSS-SECTION OF A TYPICAL TRUCK BOLSTER

FIG. C-2B. AREA MOMENTS OF INERTIA

FIG. C-2C. TRANSVERSE SHEAR

FIG. C-2D. BENDING MOMENT DIAGRAM
Putting $a = 6.75''$, $b = 28.75''$, $c = 38.75''$, $A_1 = 37.3$ in$^2$, $A_2 = 46.0$ in$^2$, $A_3 = 51.8$ in$^2$, $I_1 = 501.57$ in$^4$, $I_2 = 811.3$ in$^4$, $I_3 = 1431.3$ in$^4$ and assuming for steel $E = 30\times10^6$ psi, $G = 12\times10^6$ psi, $K = .12$ in the last equation and after some evaluation,

$$\Delta P = .3395 \times 10^{-6}$$

Bending stiffness of the bolster

$$K_{Bol} = \frac{P}{\Delta P} = 2,945,000 \text{ lb/in.}$$

**Bolster Bending Stiffness Added in Series with the Suspension Springs**

We have modelled the suspension group per truck as 4 springs each with stiffness $10518$ lb/in. Incorporating the bending effect of the bolster, the stiffness value for the suspension springs in the model becomes

$$K_{truck} = 1/(1/2,945,000) + (1/4\times10518) = 41666.67 \text{ lb/in.}$$

and the stiffness for each spring,

$$K = K_{truck}/4 = 10420 \text{ lb/in.}$$

With the bolster bending stiffness considered, the suspension spring stiffness is reduced by approximately 1%. Hence, based on this analysis, the bolster bending mode is not too significant for this particular truck type.

**C-4. Descriptive Data for a 70-Ton Box Car**

<table>
<thead>
<tr>
<th><strong>WEIGHTS AND INERTIAS</strong></th>
<th><strong>SYMBOLS</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Total car weight on rail</td>
<td>$W$</td>
</tr>
<tr>
<td>Loaded car body weight</td>
<td>$W_1*$</td>
</tr>
<tr>
<td>Bolster weight per truck</td>
<td>$W_2$, $W_3$</td>
</tr>
<tr>
<td>Side frame axle and wheel set weight per truck</td>
<td>$W_4$, $W_5$</td>
</tr>
<tr>
<td>Freight element weight</td>
<td>$FM_1g$</td>
</tr>
<tr>
<td>Car body rotational inertia about longitudinal axis</td>
<td>$I_{Y1}$</td>
</tr>
<tr>
<td>Car body rotational inertia about lateral axis</td>
<td>$I_{X1}$</td>
</tr>
<tr>
<td>Car body rotational inertia about vertical axis</td>
<td>$I_{Z1}$</td>
</tr>
</tbody>
</table>

$*W_1 = m_1g$, where $g = 386.4$ in/sec$^2$
<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truck bolster rotational inertia about longitudinal axis</td>
<td>12,000 lb in sec²</td>
</tr>
<tr>
<td>Truck bolster rotational inertia about lateral axis</td>
<td>2,640 lb in sec²</td>
</tr>
<tr>
<td>Truck bolster rotational inertia about vertical axis</td>
<td>12,000 lb in sec²</td>
</tr>
<tr>
<td>Axle-side frame-wheel set rotational inertia about longitudinal axis</td>
<td>19,200 lb in sec²</td>
</tr>
<tr>
<td>Axle-side frame-wheel set rotational inertia about lateral axis</td>
<td>120,000 lb in sec²</td>
</tr>
<tr>
<td>Axle-side frame-wheel set rotational inertia about vertical axis</td>
<td>120,000 lb in sec²</td>
</tr>
<tr>
<td>DIMENSIONS</td>
<td></td>
</tr>
<tr>
<td>Loaded car body center of gravity height above rail</td>
<td>98.5 in</td>
</tr>
<tr>
<td>Loaded car body center of gravity height above center plate</td>
<td>72.5 in</td>
</tr>
<tr>
<td>Center plate radius</td>
<td>7 in</td>
</tr>
<tr>
<td>Side bearing spacing from car centerline</td>
<td>25 in</td>
</tr>
<tr>
<td>Spring group spacing from car centerline</td>
<td>39 in</td>
</tr>
<tr>
<td>Side bearing clearance (static)</td>
<td>1/4 in</td>
</tr>
<tr>
<td>Spring travel to solid</td>
<td>3.69 in</td>
</tr>
<tr>
<td>Bolster gib clearance</td>
<td>0.375 in</td>
</tr>
<tr>
<td>Truck center distance</td>
<td>39.5 ft</td>
</tr>
<tr>
<td>Axle centers in each truck</td>
<td>68 in</td>
</tr>
<tr>
<td>Distance between rail surface variations</td>
<td>39 ft</td>
</tr>
<tr>
<td>Rail surface variation, max. vertical changes, rocking mode</td>
<td>3/4 in</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loaded car body center of gravity height above rail</td>
<td>98.5 in</td>
</tr>
<tr>
<td>Loaded car body center of gravity height above center plate</td>
<td>72.5 in</td>
</tr>
<tr>
<td>Center plate radius</td>
<td>7 in</td>
</tr>
<tr>
<td>Side bearing spacing from car centerline</td>
<td>25 in</td>
</tr>
<tr>
<td>Spring group spacing from car centerline</td>
<td>39 in</td>
</tr>
<tr>
<td>Side bearing clearance (static)</td>
<td>1/4 in</td>
</tr>
<tr>
<td>Spring travel to solid</td>
<td>3.69 in</td>
</tr>
<tr>
<td>Bolster gib clearance</td>
<td>0.375 in</td>
</tr>
<tr>
<td>Truck center distance</td>
<td>39.5 ft</td>
</tr>
<tr>
<td>Axle centers in each truck</td>
<td>68 in</td>
</tr>
<tr>
<td>Distance between rail surface variations</td>
<td>39 ft</td>
</tr>
<tr>
<td>Rail surface variation, max. vertical changes, rocking mode</td>
<td>3/4 in</td>
</tr>
</tbody>
</table>
Height of center plate above bolster center of gravity  
4.5 in  \( R_2 \)

Longitudinal distance between car body center of gravity and centerline of front bolster  
237 in  \( D_1 \)

Longitudinal distance between bolster center of gravity and suspension spring  
6 in  \( D_2 \) to \( D_5 \)

Car body center of gravity above bolster center of gravity  
74.4 in  \( B \)

STIFFNESSES AND DAMPING CHARACTERISTICS

- Suspension spring vertical stiffness  
  10420 lb/in  \( K_{13} \) to \( K_{20} \)

- Suspension spring lateral stiffness  
  4425 lb/in  \( K_{13L} \) to \( K_{20L} \)

- Center plate and side bearing stiffness  
  666,000 lb/in  \( K_1, K_2, K_7, K_8 \)

- Gib spring stiffness  
  666,000 lb/in  \( K_{GIB} \)

- Bottoming spring stiffness  
  666,000 lb/in  \( K_{BOM} \)

- Bolster and truck torsional spring stiffness  
  6,730,425 lb/rad.  \( KT_{24}, KT_{35} \)

- Bolster and truck pitching spring stiffness  
  4,200,000 lb/rad.  \( KP_{24}, KP_{35} \)

- Freight cushioning stiffness  
  200 lb/in  \( K_F \)

- Freight cushion damping coefficient  
  18 lb sec/in  \( C_F \)

- Track vertical stiffness  
  105,000 lb/in  \( K_{21} \) to \( K_{28} \)

- Track lateral stiffness  
  70,000 lb/in  \( K_{21L} \) to \( K_{28L} \)

C-5. Track Input Equations

Two types of track profiles are currently being used for the computer simulations of rocking and bounce modes.

For the rocking mode, a half-staggered, rectified sine wave is used as the track profile. The equations which describe the track inputs are (refer to Fig. C-3a) as follows:

(a) For vertical track profiles:
(a) For vertical track profiles:

\[
V_1 = S \sin (\omega t) \\
V_2 = S \sin (\omega t - \phi_1) \\
V_3 = S \sin (\omega t - \phi_2) \\
V_4 = S \sin (\omega t - \phi_2 - \phi_1) \\
V_5 = S \sin (\omega t - \phi_3) \\
V_6 = S \sin (\omega t - \phi_4) \\
V_7 = S \sin (\omega t - \phi_5) \\
V_8 = S \sin (\omega t - \phi_5 - \phi_1)
\]

where \( \phi_1 = \pi/2, \phi_2 = \pi B_1/RL, \phi_3 = \pi D/RL, \phi_4 = \phi_1 + \phi_3, \phi_5 = \phi_2 + \phi_3 \) and,

\[
\omega = \pi \text{VEL/RL},
\]

RL = rail length,

B_1 = Distance between axle centers,

S = Maximum rail surface variations,

D = Truck center distance

(b) The corresponding lateral rail profiles adopted presently are:

\[
R_9 = V_1/2 \\
R_{10} = 0 \\
R_{11} = V_3/2 \\
R_{12} = 0 \\
R_{13} = V_5/2 \\
R_{14} = 0 \\
R_{15} = V_7/2 \\
R_{16} = 0
\]

For the bounce mode, in order to decouple the responses somewhat, the two tracks are zero staggered, i.e. the rail joints on the opposite tracks are in phase, the following equations describe the track profiles:

(a) For vertical track profiles (refer to Fig. C-3b):
FIG. C-3A. ROCKING MODE VERTICAL TRACK PROFILE - HALF-STAGGERED

FIG. C-3B. BOUNCE MODE VERTICAL TRACK PROFILE - RAIL JOINTS IN PHASE

FIG. C-3c. LATERAL TRACK PROFILE
\begin{align*}
V_1 &= S|\sin(\omega t)| \\
V_2 &= V_1 \\
V_3 &= S|\sin(\omega t - \phi_2)| \\
V_4 &= V_3 \\
V_5 &= S|\sin(\omega t - \phi_3)| \\
V_6 &= V_5 \\
V_7 &= S|\sin(\omega t - \phi_5)| \\
V_8 &= V_7
\end{align*}

(b) The tracks are assumed to be parallel and no lateral variations.
REFERENCES AND BIBLIOGRAPHY


1-2. Publication of Association of American Railroads: "Loading Pamphlets" and "General Information Series".


