GENERAL MODELS FOR LATERAL STABILITY ANALYSES OF RAILWAY FREIGHT VEHICLES

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# General Models for Lateral Stability Analyses of Railway Freight Vehicles

**Title and Subtitle:**
GENERAL MODELS FOR LATERAL STABILITY ANALYSES OF RAILWAY FREIGHT VEHICLES

**Abstract:**
This report presents the development of general analytical models for use in exploring the nature of freight car hunting and for finding means of controlling the hunting behavior.

These models result from one aspect of the "Freight Car Dynamics" research project conducted by Clemson and Arizona State Universities in cooperation with the Association of American Railroads. This effort is directed at developing models to describe rail vehicle curving behavior and response to track irregularities in addition to those models presented in this report that deal with hunting.

First, a model of a wheelset with lateral, yaw, and axle torsional degrees of freedom is developed. Secondly, two such wheelsets are included in a general model of a 9 degree of freedom truck that has lateral, yaw, and warp degrees of freedom in addition to relative lateral and yaw motions of the wheelsets with respect to the truck frame. By suitable choices of primary suspension elements, this general model may be specialized to become (1) a roller-bearing freight truck, (2) a plain-bearing freight truck, (3) a roller-bearing truck with primary suspension elements, (4) a passenger truck, (5) a generic model of a freight truck with interconnected wheelsets, or (6) a rigid truck. Finally, two such truck models are combined with a car body that has lateral, yaw, and roll rigid body degrees of freedom plus two degrees of freedom that serve to approximate the first lateral bending and torsional modes. For all three models, the effects of design parameters on the critical speed for hunting are examined.

**Key Words:**
Train-Track Dynamics
Rail Vehicles and Components
Freight Car Design
Stability
Hunting

**Supplementary Notes:**
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# TABLE OF CONTENTS

## LIST OF TABLES

<table>
<thead>
<tr>
<th>List of Tables</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>v</td>
</tr>
</tbody>
</table>

## LIST OF FIGURES

<table>
<thead>
<tr>
<th>List of Figures</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>v</td>
</tr>
</tbody>
</table>

## CHAPTER:

1. **INTRODUCTION** ........................................... 1

2. **MODELING APPROACH** ...................................... 5

   - Introduction ......................................... 5
   - General Considerations ............................... 5
   - Railway Vehicle Modeling ............................ 8
     - Wheel/Rail Contact Forces ......................... 8
     - Wheel/Rail Contact Geometry ...................... 15
   - Suspension ............................................ 23
   - Car Body/Bolster Connection ....................... 40
   - Vehicle Component Flexibility ..................... 42

3. **TORSIONALLY FLEXIBLE WHEELSET STUDY** ............... 48

   - Introduction ......................................... 48
   - Description of Model ............................... 48
   - Equations of Motion ............................... 49
   - Results ............................................. 52
     - Modes of Oscillation ............................ 54
     - Effects of Design Parameters on Stability of Flexible Wheelset Model .......................... 63
   - Conclusions ......................................... 73

4. **SUSPENSION STUDY** ...................................... 75

   - Introduction ......................................... 75
   - Description of Model ............................... 75
   - Equations of Motion ............................... 80
   - Results ............................................. 81
     - Freight Truck Study ............................ 81
     - Wheelset Interconnection Study ............... 102
   - Conclusions ......................................... 112

5. **CAR BODY FLEXIBILITY STUDY** .......................... 115

   - Introduction ......................................... 115
   - Description of Model ............................... 115
   - Equations of Motion ............................... 119
   - Results ............................................. 119
   - Conclusions ......................................... 142
TABLE OF CONTENTS (Cont'd.)

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. SUMMARY AND CONCLUSIONS</td>
<td>145</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>151</td>
</tr>
<tr>
<td>A. Derivation of Equations of Motion for Flexible Wheelset Model</td>
<td>152</td>
</tr>
<tr>
<td>B. Equations of Motion for 11 DOF Model</td>
<td>184</td>
</tr>
<tr>
<td>C. Equations of Motion for Complete Vehicle Model</td>
<td>192</td>
</tr>
<tr>
<td>LIST OF SYMBOLS</td>
<td>209</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>217</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Nominal Design Parameters for Wheelset Model</td>
<td>51</td>
</tr>
<tr>
<td>4.1</td>
<td>Nominal Design Parameters for Eleven Degree-of-Freedom (11 DOF) Model</td>
<td>82</td>
</tr>
<tr>
<td>4.2</td>
<td>Response Parameters of 11 DOF Model with Different Truck Configurations</td>
<td>99</td>
</tr>
<tr>
<td>4.3</td>
<td>Changes to 11 DOF Model Nominal Parameters for Wheelset Interconnection</td>
<td>105</td>
</tr>
<tr>
<td>5.1</td>
<td>Nominal Design Parameters for Complete Vehicle Model</td>
<td>121</td>
</tr>
<tr>
<td>A.1</td>
<td>Matrix Elements for Wheelset Model</td>
<td>182</td>
</tr>
<tr>
<td>B.1</td>
<td>Matrix Elements for 11 DOF Model</td>
<td>185</td>
</tr>
<tr>
<td>C.1</td>
<td>Matrix Elements for Complete Vehicle Model</td>
<td>193</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Normal Forces Acting on a Wheelset</td>
<td>14</td>
</tr>
<tr>
<td>2.2</td>
<td>Effect of Wheel Wear on Describing Functions for Wheel/Rail Contact Angle Difference</td>
<td>17</td>
</tr>
<tr>
<td>2.3</td>
<td>Wheel Profiles from a 70 Ton Hooper Car</td>
<td>18</td>
</tr>
<tr>
<td>2.4</td>
<td>Rail Head Profiles</td>
<td>19</td>
</tr>
<tr>
<td>2.5</td>
<td>Example Wheel Profiles for Wheels with 85,000 miles of Wear</td>
<td>21</td>
</tr>
<tr>
<td>2.6</td>
<td>Conventional North American Freight Truck with Roller Bearings</td>
<td>26</td>
</tr>
<tr>
<td>2.7</td>
<td>Conventional North American Freight Truck with Plain Bearings</td>
<td>27</td>
</tr>
<tr>
<td>2.8</td>
<td>Roller Bearing Configuration</td>
<td>29</td>
</tr>
<tr>
<td>2.9</td>
<td>Secondary Suspension Configuration for a Typical Freight Truck</td>
<td>31</td>
</tr>
<tr>
<td>3.1</td>
<td>Schematic of Wheelset Model</td>
<td>50</td>
</tr>
<tr>
<td>3.2</td>
<td>Effect of Axle Diameter on Axle Torsional Stiffness</td>
<td>53</td>
</tr>
<tr>
<td>3.3</td>
<td>Variation of Eigenvalues of Wheelset Oscillatory Modes with Axle Torsional Stiffness for V = 100 mph</td>
<td>55</td>
</tr>
<tr>
<td>3.4</td>
<td>Variation of Eigenvalues of Wheelset Oscillatory Modes with Axle Torsional Stiffness for V = 300 mph</td>
<td>56</td>
</tr>
<tr>
<td>3.5</td>
<td>Variation of Eigenvalues of Wheelset Oscillatory Modes with Speed for Several Values of Axle Torsional Stiffness</td>
<td>58</td>
</tr>
<tr>
<td>3.6</td>
<td>Effect of Axle Torsional Stiffness on Stability of the Wheelset Model</td>
<td>59</td>
</tr>
<tr>
<td>3.7</td>
<td>Effect of Axle Torsional Stiffness on Stability of the Wheelset Model for Different Values of Wheel Conicity</td>
<td>64</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.8</td>
<td>Effect of Axle Torsional Stiffness on Stability of the Wheelset Model for Different Values of Primary Yaw Stiffness</td>
<td>68</td>
</tr>
<tr>
<td>3.9</td>
<td>Effect of Axle Torsional Stiffness on Stability of the Wheelset Model for Different Values of Creep Coefficients</td>
<td>70</td>
</tr>
<tr>
<td>3.10</td>
<td>Effect of Axle Torsional Stiffness on Stability of the Wheelset Model for Different Values of Axle Torsional Damping</td>
<td>72</td>
</tr>
<tr>
<td>4.1</td>
<td>Schematic of 11 DOF Model-Equilibrium Configuration (Plan View)</td>
<td>77</td>
</tr>
<tr>
<td>4.2</td>
<td>Schematic of 11 DOF Model-Disturbed Configuration (Plan View)</td>
<td>78</td>
</tr>
<tr>
<td>4.3</td>
<td>Schematic of 11 DOF Model (Front View)</td>
<td>79</td>
</tr>
<tr>
<td>4.4</td>
<td>Effect of Primary Suspension Stiffness on Critical Speed of the 11 DOF Model (Nominal Warp Stiffness)</td>
<td>84</td>
</tr>
<tr>
<td>4.5</td>
<td>Effect of Primary Lateral Damping on Critical Speed of the 11 DOF Model</td>
<td>87</td>
</tr>
<tr>
<td>4.6</td>
<td>Frequency vs. Wheelset Lateral Displacement of Different Coefficients of Friction: Plain Bearing Truck Configuration</td>
<td>88</td>
</tr>
<tr>
<td>4.7</td>
<td>Effect of Warp Stiffness on Critical Speed of the 11 DOF Model for the Plain Bearing Truck Configuration</td>
<td>92</td>
</tr>
<tr>
<td>4.8</td>
<td>Effect of Primary Lateral Stiffness on Critical Speed of the 11 DOF Model</td>
<td>93</td>
</tr>
<tr>
<td>4.9</td>
<td>Effect of Primary Suspension Stiffness on Critical Speed of the 11 DOF Model</td>
<td>95</td>
</tr>
<tr>
<td>4.10</td>
<td>Frequency and Damping Ratio vs. Speed for Different Truck Configurations on 11 DOF Model</td>
<td>97</td>
</tr>
<tr>
<td>4.11</td>
<td>Effect of Axle Torsional Stiffness on Critical Speed of Nominal 11 DOF Model</td>
<td>101</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>4.12</td>
<td>Schematic of Interconnected Wheelset Model (Plan View)</td>
<td>103</td>
</tr>
<tr>
<td>4.13</td>
<td>Effect of Shear Stiffness on the Critical Speed of the 11 DOF Model with Interconnected Wheels for Different Values of Bending Stiffness</td>
<td>106</td>
</tr>
<tr>
<td>4.14</td>
<td>Mode Shape of Least-Damped Mode of 11 DOF Model with Interconnected Wheelsets</td>
<td>110</td>
</tr>
<tr>
<td>5.1</td>
<td>Schematic of Complete Vehicle Model</td>
<td>117</td>
</tr>
<tr>
<td>5.2</td>
<td>Schematic of Flexible Car Body Model (Plan View)</td>
<td>118</td>
</tr>
<tr>
<td>5.3</td>
<td>Modified Heumann Wheel Profile</td>
<td>126</td>
</tr>
<tr>
<td>5.4</td>
<td>Effect of Car Body Stiffness on Critical Speed of the Complete Vehicle Model (Hopper Car with New Wheels)</td>
<td>128</td>
</tr>
<tr>
<td>5.5</td>
<td>Least-Damped Modes for Flexible Hooper Car Configurations (New and Heumann Wheel Profiles)</td>
<td>130</td>
</tr>
<tr>
<td>5.6</td>
<td>Least-Damped Modes for Rigid Hooper Car Configurations with New Wheels</td>
<td>131</td>
</tr>
<tr>
<td>5.7</td>
<td>Effect of Car Body Stiffness on Critical Speed of the Complete Vehicle Model (Hopper Car with Heumann Wheel Profiles)</td>
<td>132</td>
</tr>
<tr>
<td>5.8</td>
<td>Least-Damped Modes for Rigid Hooper Car Configurations with Heumann Wheel Profiles</td>
<td>135</td>
</tr>
<tr>
<td>5.9</td>
<td>Effect of Car Body Stiffness on Critical Speed of the Complete Vehicle Model (Flat Car with New Wheels)</td>
<td>137</td>
</tr>
<tr>
<td>5.10</td>
<td>Effect of Car Body Centroid Height on Critical Speed of Complete Vehicle Model (Heavy Flat Car with New Wheels)</td>
<td>141</td>
</tr>
<tr>
<td>A.1</td>
<td>Axis Systems for Wheelset Model</td>
<td>153</td>
</tr>
<tr>
<td>A.2</td>
<td>Forces and Moments Acting on Wheelset Model</td>
<td>158</td>
</tr>
<tr>
<td>A.3</td>
<td>Forces and Moments Acting on Wheelset Model (Right Side View)</td>
<td>159</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>A.4</td>
<td>Axis Systems at Wheel/Rail Contact Points</td>
<td>161</td>
</tr>
<tr>
<td>A.5</td>
<td>Location of Wheel and Rail Contact Points for Varying Wheelset Lateral Displacement</td>
<td>178</td>
</tr>
</tbody>
</table>
CHAPTER 1. INTRODUCTION

Background

The hunting behavior of rail freight vehicles has caused major problems for North American railroads. Heavy wear of components such as wheels and bolsters, track damage and certain derailments are attributed to freight truck hunting. This report describes general analytical models developed for use in exploring the nature of freight car hunting and for finding means of controlling the hunting behavior.

These general analytical models described result from one aspect of the "Freight Car Dynamics" research project conducted by Clemson and Arizona State Universities in cooperation with the Association of American Railroads. This effort, sponsored by the Federal Railroad Administration under contract DOT-OS-40018, has the broad objective of developing the background and mathematical tools that rail vehicle designers and operators need to eliminate present freight car dynamic problems. This effort is directed at problems due to rail vehicle curving behavior and response to track irregularities as well as to the problems of freight car hunting.

The following steps are being carried out in this research project. (1) development of mathematical models for rail vehicle hunting, lateral response to track irregularities, and curving behavior; (2) validation of these models with experimental data from freight car tests carried out by the Association of American Railroads on the Union Pacific Railroad; (3) utilization of the models to examine, in a preliminary fashion, current vehicle and track maintenance procedures, and (4) exploration of model behavior, possibly resulting in suggestions for modifications of current freight car trucks and/or development of new concepts for freight truck design.
General Mathematical Models

This document presents the most general mathematical models developed in the first step above to study freight car hunting. Parametric studies using these models verified that a simpler mathematical model may be used for studying the special case of the behavior of cars equipped with the three piece, roller bearing freight truck. This model will be described, and an extensive parameter study presented in a subsequent report. A previous report describes analytical techniques for characterizing the wheel/rail geometry [1], and subsequent interim reports will document mathematical models of freight car behavior in curving and in forced response to track irregularities.

The general analytical model described in this report can be used to describe a variety of trucks or bogies through appropriate choice of parameter values. For example, the following configurations can be modeled: plain bearing freight trucks, roller bearing freight trucks, freight trucks with primary suspension elements between sideframes and wheelset, trucks with rigid frames and primary suspensions, and rigid or flexible frame trucks with interconnected wheelsets and primary suspensions. In addition, this model can be used to investigate first order effects on vehicle lateral stability of car body lateral bending and torsional flexibility as well as axle torsional flexibility and independently rotating wheels. The model has twenty-three degrees of freedom consisting of lateral, yaw, and torsional motions of each of four wheelsets; lateral, yaw, and warping motions for each of two trucks; lateral, yaw, and roll rigid-body motions of the car body; and, approximate representations of the first lateral bending and car body torsional modes.
Although not discussed further in this report, simpler models can be derived from the general twenty-three degree-of-freedom model. These models are: (1) a nineteen degree-of-freedom model having all the degrees-of-freedom of the general model with the exception of axle torsion for the four axles, (2) a seventeen degree-of-freedom model having lateral and yaw degrees of-freedom for each axle, lateral, yaw, and warp for each truck, and lateral, yaw, and roll for the car body. These models have been used in design studies where many computer runs have been needed and the axle torsion and/or car body flexibility effects are not important.

**Parametric Studies**

Parametric studies were carried out with the general mathematical models in order to check the range of validity of the assumptions made for other existing models with fewer degrees of freedom (e.g., the assumptions of a rigid car body and torsionally rigid axles, "ball-joint" wheelset-side-frame connections for roller bearing trucks, etc.) and to examine the stability of various configurations of existing freight car designs and completely new designs incorporating features such as primary suspensions, torsionally flexible wheelsets, and wheelset interconnections.

These parameter studies were not extensive for the case of the vehicle with roller bearing trucks as that study will be presented in a subsequent report.

A two part study was conducted for an eleven degree of freedom, half car model. In the first part, the effects on stability were examined of adding primary suspension and of varying warp stiffness and axle torsional stiffness for the freight truck. In the second part, variations in bending and shear stiffness between wheelsets for a generic truck model with wheelset interconnection were studied.
For the general twenty-three degree of freedom model, studies were conducted to investigate the effects on stability of: (1) variations in car body stiffness for cars with new and curved wheel profiles, (2) the amount of lading, and, (3) variations in the height of the car body centroid. The two types of car bodies considered are the L&N 80 ton open hopper car and an ACF 70 ton, 89 ft TOFC or flat car. Nominal roller bearing trucks were used in these studies.

Outline of Report

The analytical models and parametric studies outlined above are discussed in the following chapters. The process of modeling dynamic systems, and the specific details of analyzing rail vehicles are discussed in Chapter 2. A three degree of freedom model for a wheelset with a torsionally flexible axle is presented in Chapter 3 with results from a parameter study of variations in wheelset parameters. A suspension system parametric study for an eleven degree of freedom, half car model is discussed in Chapter 4. The most general, twenty three degree of freedom analysis was used for the car body flexibility study reported in Chapter 5. The conclusions concerning rail vehicle analysis and design obtained from this study are covered in Chapter 6. The most significant findings of the parametric studies on freight vehicle stability are discussed and summarized in Chapter 6. Details of the derivations of the wheelset equations of motion are described in Appendix A. The derivations of the equations for the general eleven degree of freedom, half car model and the twenty three degree of freedom general car model are presented in [2] and the equations of motion listed in Appendices B and C, respectively.

The work described in this report has been published in a slightly different form as Mr. Jeffrey A. Hadden's Master's Thesis [2].
CHAPTER 2. MODELING APPROACH

Introduction

The process of modeling a general dynamic system together with applications of this process to the modeling of a railway vehicle is discussed in this chapter. The modeling task as applied to a railway vehicle can be addressed under five areas, which are:
- Wheel/Rail Contact Forces
- Wheel/Rail Contact Geometry
- Suspension Elements
- Car Body/Bolster Connection
- Vehicle Component Flexibility

Methods for modeling the vehicle are discussed and evaluated. The assumptions which are necessary in developing a vehicle model are discussed. These assumptions determine the applicability of the model and the implications of these assumptions must be fully understood before the results obtained from the model can be interpreted and used.

General Considerations

To model a dynamic system, it is desirable to develop the simplest credible model which describes the motions of the system. The "simplicity" of the model is determined to a large extent by the number of degrees-of-freedom. The number of degrees-of-freedom is a result of the approximations made in establishing the physical model of the actual system. Adding degrees-of-freedom to a model usually increases the number of parameters needed to describe the actual system. Typically, these parameters are suspension connections which may include springs, viscous dampers and coulomb friction
elements. To model the connections accurately, the force versus deflection and/or force versus velocity characteristics of the suspension connections must be known. If these are not known, they must be "guessed", and the increased complexity of the model does not increase its validity. Even if the characteristics of the suspension connections are known, it can become more difficult with an increasing number of degrees-of-freedom to interpret the behavior of the model. From an economic point of view, more computer time is usually required to analyze the behavior of the model as degrees-of-freedom are added.

The "credibility" of the model is determined in part by whether the degrees-of-freedom used will describe the system to the accuracy that is desired. For example, a model with a minimum number of degrees-of-freedom, which implies some general approximations, could give a generic representation of the system from which qualitative trends may be studied. Alternatively, it may be necessary to use a larger number of degrees-of-freedom (and thus a more exact representation of the system) to model adequately a particular system so that quantitative, numerical results can be obtained and directly applied to an analysis of the physical system.

The validity of a model can be checked by comparing it to existing models which have been validated by experiment and by comparing its behavior to the behavior of the actual system. For example, "limiting cases" may be prescribed for a more complex model such that it is mathematically similar to a simpler existing model. Then, the behavior of the models should be similar. After the model has been checked against existing models, it should be
checked against experimental results. If necessary, the model should be modified so that its behavior represents the observed behavior of the actual system.

The methods of analysis used to perform studies with a model depend largely on the purpose of the studies. A variety of such studies are reviewed in reference [3]. The response of the model to prescribed inputs or forces or known initial conditions can be studied by integrating the differential equations which describe the model. The behavior of the model is then interpreted from displacement time-histories. Both linear and highly nonlinear models can be integrated directly by numerical methods or by analog integration.

For linear models, various solution methods can be used to examine dynamic behavior. Frequency response methods are used to study the response to sinusoidal inputs. When power spectral density functions are available, the response to random inputs can be studied by methods similar to those for studying frequency response. Eigenvalue/eigenvector analyses are used to study the unforced response, or stability, or a system. Response characteristics such as frequency, damping factor and mode shapes are obtained from these methods. These methods are limited to linear models, and nonlinear models must be linearized by one of several techniques in order to use them.

As this report is concerned with the stability of linearized models, an eigenvalue/eigenvector method is used. This solution method is described in the Appendices.

Thus, there are three phases to be performed when using a model to study the behavior of a physical system. First, the simplest
credible model is developed. This process is strongly influenced by the available information about the characteristics of the actual system and the purpose for which it will be used. Then, the model can be validated against existing models and by comparing its behavior with the observed behavior of the actual system. The ultimate test for validity of any model is how well it approximates the actual physical system. After the model is validated, various methods can be used to study the behavior of the model. The method used is influenced by the type of model (i.e., linear or nonlinear), the objectives of the studies and the ultimate use of the results of the studies.

**Railway Vehicle Modeling**

As previously stated, one can consider five general areas when modeling a railway vehicle. Each area will now be discussed. This report is concerned with the stability of freight cars. Therefore, discussions will concentrate on the design of freight car components.

**Wheel/Rail Contact Forces**

Neglecting aerodynamic forces and forces exerted by shifting passengers or cargo, the forces acting on a railway vehicle may be exerted by other vehicles, and by the rails on the wheels. Except in traction and braking studies, almost all lateral dynamics studies have assumed the vehicle is traveling at a constant velocity and is not coupled to another vehicle. In other words, it is commonly assumed that the vehicle is coasting alone. This assumption
is justified by Blader and Kurtz [4], who state that the stability of a group of vehicles is never worse than the stability of a single vehicle. They compared results obtained from a single freight car model with that of a freight train model. They found that the critical speed obtained for the single uncoupled freight car model was virtually identical to that of the freight train model.

Forces applied to the vehicle from the rails are normal forces and creep forces. This report studies the stability of vehicle models and the rails are assumed perfectly smooth and rigid. A more refined stability model would include effects of flexible rails. Wickens [5] showed that increasing track flexibility is destabilizing. He modeled the flexibility of the track by springs connected to the rails so that the rails can move laterally and roll together. Wickens found that the flexibility of typical track has a small effect on the stability of a vehicle. To study the dynamic response of a model, as opposed to the stability, both random rail irregularities and rail flexibility can be included. Thus, depending on the purpose of the studies, random rail irregularities and rail flexibility may or may not be considered.

Among those who conducted studies using various rail models is Law [6], who considered the response of a nonlinear wheelset to random lateral rail irregularities. He also modeled the lateral flexibility of the rails using simple deadband springs. The flexible rail model provides a gross approximation to the occurrence of flange contact. Cooperrider [7] investigated the response of a seven degree-of-freedom truck to random lateral rail irregularities.
Ahlbeck et al [8] discuss modeling techniques used to develop realistic rail inputs to a fourteen degree-of-freedom vehicle/track model.

**Creep Forces**

Creep forces are the tangential forces generated between the wheels and rails due to a difference in the rates of strain, or creepage, in the contact region. Creepage can be defined as the relative velocity between the wheels and rails at contact, normalized by the forward velocity of the wheelset. If the rails are assumed rigid, then the creepage of a wheel is the velocity of a point in the wheel at the contact point divided by the forward velocity of the axle. If both the wheels and the rails are considered rigid, then no creepage exists and a state of either pure rolling or pure sliding exists. The force versus relative velocity relationship for the case of rigid wheels rolling on rigid rails is shown in the sketch below and is the classical coulomb or dry friction characteristic.

![Coulomb Friction](image)

Carter [9] was the first to investigate the problem of creep. He developed the exact solution for the longitudinal creep of two
cylinders in rolling contact. Creep forces are linearly proportional to the creepage up to a point of limiting adhesion, beyond which gross sliding of the wheel on the rail occurs. A general relationship between creep force and creepage is shown in the sketch below.

![Creep Force vs. Creepage](image)

The importance of creep forces to the dynamic response, and particularly to the stability of railway vehicles has been acknowledged and investigated by several persons during the past half century. Hobbs [10] has surveyed the research done on the phenomenon of creep. The most recent advances in creep theory have been made by Kalker, who has developed linear and nonlinear creep theories including spin effects [11, 12]. Kalker's linear theory is applied in this thesis.

In linear creep theory, Hertzian contact is assumed to exist between the wheels and rails. The creep forces are defined as the product of the creepages and "creep coefficients". The creep coefficients are defined for each wheel and are functions of the normal force exerted between the wheel and rail, the modulus of elasticity, and the ellipticity of the contact ellipse predicted by Hertz's contact theory [13].
The magnitudes of creep coefficients estimated in experiments have varied from Kalker's predicted values by as much as fifty percent for contaminated surfaces [14]. For uncontaminated surfaces, some experimental results have shown good agreement for Kalker's predicted values. For example, as reported by Hobbs [10], Müller performed experiments with a full scale wheelset on a short length of track for varying wheel loads and for dry and wet surface conditions. His results for dry (and extremely clean) surface conditions showed good agreement with Kalker's theoretical predictions. The measured creep coefficients for wet surface conditions were up to 30 percent less than for dry surface conditions for the low value of wheel load (4,400 lb.), and 12.5 percent less for the highest wheel load (22,000 lb.).

Surface contamination may be one reason for discrepancies between measured creep coefficients and those predicted by theory. Other reasons may be work hardening of the surfaces with use and vibration.

Because of the wide range of wheel and rail conditions and track environment, it is very difficult to model accurately creep forces. Thus, only "reasonable" approximations of the magnitudes of the creep coefficients can be made. Alternatively, it may be useful to vary the creep coefficients over the range of values which may be encountered in practice.

In the vast majority of railway vehicle dynamic models developed to date, the creep forces are modeled according to linear creep theory. This is done when the models are used for stability studies.
An obvious limitation of the linear creep model is that slipping of the wheel on the rail is not modeled while it can occur in practice. However, for small displacements and for use in stability investigations, the linear creep model can be a reasonable approximation to actual creep phenomenon. However, one must be careful when predicting the behavior of an actual rail vehicle from its model with linear creep. One set of values for the creep coefficients cannot be representative of the range of wheel/rail conditions encountered in practice.

Normal Forces or Gravitational Stiffness

Due to the taper, or conicity, of the wheels, the normal forces acting on a wheelset have both lateral and vertical components, as shown in Figure 2.1. If the wheelset is yawed in the horizontal plane, a moment acts about the wheelset center of gravity due to the lateral component of the normal forces. The net effect of the lateral components of the normal forces acting on a wheelset is often referred to as the "gravitational stiffness effect". The gravitational stiffness force is a function of the axle load, the wheelset roll angle, and the difference in contact angles between the left and right wheels.

The combined effects of the gravitational stiffness and lateral/spin creep may be expressed as the product of axle load and a constant for linearized equations. For light axle load, this is a small term and may be neglected. As axle loads increase, this term becomes increasingly important. For axle loads typical of heavy freight cars, this effect may be strong. As Cooperrider et al [15]
FIGURE 2.1 Normal Forces Acting on a Wheelset
state, "the gravitational stiffness effect can provide more restoring force than the vehicle suspension for a heavily loaded wheelset."

The wheelset roll angle contribution to the gravitational stiffness effect is about the same for new and worn wheels. The difference in contact angles however, is strongly influenced by the condition of the wheels, and determines whether the gravitational stiffness is a stabilizing or destabilizing effect. The influence of wheel/rail contact geometry on the gravitational stiffness effect and on the stability of a rail vehicle is discussed in detail in the next section.

Wheel/Rail Contact Geometry

As discussed previously, the stability of a rail vehicle can be strongly affected by the normal forces or gravitational stiffness effect. The effect of the gravitational stiffness on stability is in turn influenced by the shape of the wheel and rail profiles and the geometry of wheel/rail contact. The contact geometry may be defined by the wheelset roll angle, the rolling radii of the wheels, and the contact angles between the wheels and the rails. These kinematic quantities are functions of the wheelset lateral displacement. Wheelset yaw displacement has a secondary effect on the kinematic quantities.

As the wheels and rails wear, their contact geometry can change drastically. As an illustration, the describing function for the difference in contact angles as a function of wheelset lateral displacement is shown for several conditions of wheel
wear in Figure 2.2. These curves (from [1] and [15]) indicate the change in the wheel/rail contact geometry with wear.

The wheel/rail geometry affects stability by the influence on gravitational stiffness and on wheel conicity. The stabilizing effect of the gravitational stiffness increases with increasing contact angle difference. The conicity of the wheels is a well known destabilizing effect and is determined by the difference in rolling radii of the wheels. It is the combined influence of the two effects that determines the effect on stability of the wheel/rail geometry. As the gravitational stiffness is a function of the axle load as well as the contact angle difference, the effect may be larger than the conicity effect for large axle loads. Cooperrider et al [15] found that the conicity and gravitational stiffness effects for a "slightly worn" wheel profile were large compared to the same effects for new wheels.

It is a relatively simple matter to model the contact geometry for new wheels and new rails, as the kinematic quantities are constants or linear functions of the wheelset lateral displacements in the tread region. As the wheels and rails wear with use, however, the profile characteristics become quite nonlinear, as can be seen in Figures 2.3 and 2.4. Further, asymmetric wear between the left and right wheels and rails is not uncommon for North American freight cars. A typical case of asymmetric wheel wear is shown in Figure 2.5. Thus, it is difficult to model the geometries of a particular set of worn wheel and rail profiles, and it is impossible to develop a model of a worn wheel or worn rail profile which is
\[ \left( \frac{\delta_L - \delta_R}{2} \right) \text{ quasi-linear} = \text{Describing Function} \times \frac{x}{\alpha} \]

**CONTACT ANGLE DIFFERENCE DESCRIBING FUNCTION**

**vs.**

**NONDIMENSIONAL WHEELSET LATERAL DISPLACEMENT.**

(WORN RAILS, NOMINAL GAUGE, \( \alpha = 29.562 \) in.)

**FIGURE 2.2** Effect of Wheel Wear on Describing Functions for Wheel/Rail Contact Angle Difference (from [15])
FIGURE 2.3-Wheel Profiles from a 70 Ton Hopper Car
FIGURE 2.4 Rail Head Profiles

ESTIMATED TOTAL TONNAGE OVER RAIL
898.2 MILLION GROSS TONS
characteristic of all worn profiles. Consequently, analytical results obtained using a model of a particular set of worn profiles cannot be representative of trends for worn profiles in general.

For linear vehicle models, effective values for the nonlinear kinematic functions for worn profiles may be obtained by linearization methods. A problem is to measure accurately the wheel/rail contact geometry in order to obtain these functions. Cooperrider et al. [1] developed analytical and experimental methods for determining the kinematic functions for given wheel and rail profiles, cant angle, and rail gauge, as functions of wheelset lateral displacement. The results obtained from using two methods showed good agreement. Thus, kinematic functions obtained by their analytical method may be used directly in nonlinear analyses (e.g., predicting limit cycle behavior). They also used quasilinearization methods to obtain describing functions for the kinematic functions. These describing functions are used to obtain Heumann wheel profile parameters in the study presented in Chapter 5.

Wickens [16] makes a valid point that "the validity of design studies performed on the assumption of purely conical profiles is questionable", and the range of effective conicities encountered in practice with worn profiles should be considered. Müller [17] suggests that a worn wheel profile which is stable with respect to wear can be selected for use. Then, vehicles may be designed with "worn" profiles to provide good dynamic performance. Thus, the profile should not wear appreciably and contact stresses will be reduced. Wickens [18] warns of the difficulty in designing
FIGURE 2.5 Example Wheel Profiles for Wheels with 85,000 Miles of Wear
In summary, the dynamic behavior of a rail vehicle is strongly influenced by the condition of the wheels and rails. For example, the speed at which sustained hunting oscillations occur may vary significantly with wheel wear. Conventional wheel profiles tend to wear rapidly and sometimes wear asymmetrically. This presents a problem for accurate rail vehicle modeling. The dynamic behavior of vehicles with new wheels is not representative of the behavior with worn wheels. Wheel wear patterns vary widely and one cannot select a set of worn profiles that are characteristic of all worn wheels. This is a dilemma that must be solved by the railroad industry.

Possible solutions to this dilemma are 1) to machine periodically worn profiles back to new profiles, 2) to design a vehicle to have acceptable dynamic performance for a range of wheel and rail profiles that might be expected in practice, or 3) to adopt Müller's suggestion to select a design for a "stable" worn wheel profile, i.e., a profile which will not change appreciably with use. The first alternative is currently practiced to a limited extend in the United States and by the Japanese National Railroad on the New Tokaido Line. The machining process, however, is costly and does not solve the problem of poor dynamic behavior due to wheel and rail wear. The second alternative, as Wickens indicates, may be very difficult to implement due to the wide range of worn profiles encountered in practice. The third alternative may be
effective in reducing wheel and rail wear and in providing consistently good vehicle performance. Redesigning or retrofitting current suspensions for a stable worn wheel profile would be a formidable and expensive task for the railroad industry, but long range benefits, such as decreased maintenance requirements, consistently good dynamic performance and increased reliability would probably justify the initial efforts and costs. "Worn" wheel profiles are used on the British Rail Advanced Passenger Train (APT) and on the prototype Scheffel truck. These vehicles are recent designs, and have not been in use long enough to determine the long-term advantages or disadvantages of using the worn profiles.

Suspension

Generally, the suspension of a railway vehicle can be defined as the mechanism which connects the car body to the rails. This mechanism consists of two or more trucks or bogies. Each truck consists of one or more axles which are connected through some type of primary suspension to a truck frame. The truck frame supports the weight of the car body through a secondary suspension system.

Truck configurations differ according to their purposes. Except for cars which carry the heaviest loads, almost all conventional single railway vehicles use two dual axle trucks. This broad category of trucks consists of passenger trucks and freight trucks. As this report is concerned mainly with the stability of freight cars, the following discussion of suspension will concentrate on freight trucks.
The parametric studies described in Chapter 4 consider some freight truck designs with features which are characteristic of passenger trucks. Thus, an overview of passenger truck designs is now presented.

**Passenger Trucks**

A passenger truck may be characterized by an effectively rigid truck frame and primary and secondary suspensions which are designed for good passenger ride comfort and good overall dynamic behavior.

The primary suspension consists of the restoring and dissipative elements which connect the wheel sets to the truck frame. For passenger trucks, these can consist of elements such as elastomer pads, steel/rubber chevrons or coil springs. These elements allow wheelset motions relative to the truck frame and help reduce vibration transmission to the car body.

The secondary suspension supports the car body either directly or through the bolster. Many types of secondary suspension configurations are used in passenger trucks. As an example, the bolster may be rigidly connected to the rest of the truck frame and some type of air springs can support the car body. In another configuration, vertical spring groups in parallel with hydraulic dampers may act between the bolster and truck frame, and the car body is supported by the bolster at a centerplate connection. (The centerplate connection is discussed in detail in the next section). Other passenger truck designs are in use.

The degrees-of-freedom which are usually modeled for passenger trucks are lateral and yaw displacements of the wheelsets, and lateral,
yaw and roll displacements of the truck frame. The bolster may have lateral, yaw, and roll displacements relative to the truck frame. Thus, it is possible to model a truck with twelve degrees-of-freedom. Six or seven degrees-of-freedom are assumed for most passenger truck models. These are wheelset and truck lateral and yaw displacements, and sometimes truck roll displacement.

**Freight Trucks**

Generally, freight truck designs are less sophisticated than passenger truck designs. Essentially, conventional freight trucks differ from passenger trucks by 1) the flexibility of the truck frame, 2) the lack of primary suspension elements between the wheelsets and truck frame, and 3) the (intentional) use of friction in parallel with springs in the secondary suspension and friction acting throughout the truck. A typical North American freight truck is shown in Figure 2.6. This figure is a helpful reference for the remainder of the chapter.

A conventional North American freight truck frame consists of two sideframes and a bolster. The ends of the bolster rest on vertical spring groups, which in turn rest in the sideframes. This three-piece frame can deform so that the wheelsets and sideframes assume a sort of parallelogram shape. This type of deformation has been called "lozengine", "parallelogramming", and "warping". Warping motion is resisted by friction between the bolster and sideframes and between the sideframes and wheelsets. This motion is limited by contact of the ends of the bolster against the side-
FIGURE 2.6 Conventional North American Freight Truck with Roller Bearings.
FIGURE 2.7 Conventional North American Freight Truck With Plain Bearings
frames (The maximum allowable relative yaw angle between the bolster and sideframes is about 2.5 degrees for conventional three-piece freight truck frames). Soft primary suspension, which provides the wheelset yaw flexibility needed for good curve negotiation, does not exist for conventional North American freight trucks (Primary suspension is found on some European trucks). Wheelset yaw flexibility is provided by the warping motion of the truck frame.

The wheelsets of conventional North American freight trucks are connected to the sideframes through either plain bearings or roller bearings. There is usually no resilient material between the bearings and sideframes. Consequently there is more restraint on relative motions between the wheelsets and truck frames for freight trucks than for passenger trucks.

Wheelset lateral motion is relatively unrestrained in plain or journal bearing boxes until the allowed clearance is taken up and the axle hits stops. Longitudinal or yaw motions of the wheelsets relative to the truck frame with plain bearings are virtually eliminated. A freight truck with plain bearings is shown in Figure 2.7.

Roller bearings permit a very small amount of wheelset lateral motion with respect to the truck frame, and permit some wheelset yaw motion with respect to the truck frame from the available clearances between the bearings and the sideframes. A close-up of the sideframe-wheelset interface on a truck with roller bearings is shown in Figure 2.8.
FIGURE 2.8 Roller Bearing Configuration
As the two bearing types constrain wheelset motions differently, the "truck hunting" motions differ for roller bearing trucks and plain bearing trucks. For roller bearing trucks, wheelset lateral and yaw motions are strongly coupled to motions of the truck frame. The hunting mode for freight cars with roller bearing trucks usually consists of large motions of the entire truck. For plain bearing trucks, lateral motions of the wheelset can be large with respect to the truck frame. The hunting mode for plain bearing trucks usually consists of large wheelset lateral motions and small motions of the truck frame and car body. The critical speeds for freight vehicles with plain bearing trucks have been observed to be about ten miles per hour lower than the critical speeds of freight vehicles with roller bearing trucks [19].

The secondary suspension on a conventional North American freight truck consists of vertical spring groups (which can "bottom") in parallel with friction plates. The spring groups rest in the sideframes and support the bolster at its ends. The design of the secondary suspension on an ASF Ride Control truck is shown in Figure 2.9 [20]. The suspension forces act to oppose lateral, vertical, and roll motions of the bolster relative to the sideframes.

The behavior of the secondary suspension groups may be influenced by the conditions at the sideframe/bearing adapter connection. Each sideframe is capable of a small amount of roll motion about the sideframe/bearing adapter connections. This roll motion can influence the behavior of the secondary suspension. In
FIGURE 2.9 Secondary Suspension Configuration for a Typical Freight Truck
the new condition, point contact exists at these connections and allows the greatest amount of sideframe roll motion. As the components wear, the sideframe/bearing adapter connections are flattened, probably decreasing or eliminating any sideframe roll motion.

Wheelset Interconnection

Much interest has been focused recently on the concept of a "self-steering truck". In light of this interest, a generic model of a truck with interconnected wheelsets was developed and is briefly analyzed in Chapter 4.

By interconnecting the wheelsets by a frame or yoke, a yaw moment and a lateral force is transmitted between wheelsets. Then, as the truck negotiates a curve, the wheelsets will tend to align themselves radially to the curve as shown in the sketch below.

![Curving Behavior of Interconnected Wheelsets](image)

As shown by Newland [21], alignment of the wheelsets in this manner can be desirable, as wheel slip and flange contact are greatly reduced during curving.
List [22] discusses conceptually the general advantages and disadvantages of rigid, passenger, freight, and "self-steering" trucks on tangent and curved track. He concludes that the self-steering truck can improve curving performance and stability. Interconnected wheelset designs are analyzed by Wickens [23], Sheffel [24], and List et al [25].

Wickens shows that wheelset interconnection is characterized by the bending stiffness and shear stiffness (corresponding to \( k_{\theta L} \) and \( k_{sL} \), respectively in the sketch shown above) between the wheelsets. Zero bending stiffness allows perfect curving, but without sufficient shear stiffness, the critical speed is zero. Wickens analyzes the stability and curving performance of vehicles with wheelset interconnection.

Scheffel analyzes the stability and curving performance of a pair of wheelsets interconnected by a lateral elastic restraint such that little or no bending stiffness is transmitted between the wheelsets. He shows that this arrangement improves curving performance and does not degrade vehicle stability. List et al discuss the results of tests and analyses of three proposed truck designs with interconnected wheelsets. They maintain that wear, operating costs, and derailments would be reduced by use of these trucks. They also claim that, with wheelset interconnection, truck hunting can be reduced or eliminated and flange contact during curving is reduced.

In the near future, Dresser Transportation Equipment Division plans to build and market a truck with interconnected wheels that was designed by List. A truck designed by Scheffel is being considered for manufacture by Standard Car Truck.
Suspension Modeling

A linearized model is an approximation to the physical system. The validity of the model is determined by the accuracy of the approximations and the assumptions. A conventional freight truck is a highly nonlinear dynamic system and the validity of the linearized freight truck model depends on how accurately the nonlinearities are approximated.

The nonlinearities that exist for a freight truck fall into the following categories: 1) nonlinear springs, 2) coulomb friction elements, and 3) clearances and stops. They can be linearized with varying degrees of accuracy.

The secondary spring groups may have "hard" or "soft" force versus deflection characteristics, as shown below in the sketches. (Secondary spring groups have typically "hard" characteristics).

Nonlinear Spring Characteristics.

Nonlinear springs can be linearized by determining an effective stiffness over a range of spring deflections. For small displacements of the model from equilibrium, the linear spring model can be very accurate. For large motions, and thus large spring deflections,
the effective stiffness approximation may not be accurate, as spring bottoming may occur as the clearances are taken up.

The simplest method to obtain a linear approximation of coulomb friction is to calculate an equivalent viscous damping coefficient [26]. To obtain the equivalent viscous damping, one must judiciously "guess" an amplitude and frequency of the relative motion between the components. Thus, the approximation is only as good as the "guesses". Further, the phenomenon of gross sliding and "sticking", which can occur for actual friction, cannot occur for equivalent viscous damping. Den Hartog [27] showed (for a single degree-of-freedom system) that unbounded forced response of a system is possible near resonance with friction damping, while this cannot occur if there is any viscous damping present.

The equivalent viscous damping linearization of coulomb friction allows linear techniques to be used to study stability and dynamic behavior. It is probably the least accurate linear model of the nonlinear elements in a freight truck in terms of predicting response to inputs for the reasons discussed previously. It should be noted however, that the energy dissipated per cycle with equivalent viscous damping is identical to that for the coulomb friction element. For stability analyses, the equivalent viscous damping model of friction can be adequate for predicting the critical speeds of sustained hunting oscillations, stability trends due to design changes, and frequencies and mode shapes of vehicle motions.

Cooperrider et al [28] used describing function and numerical optimization techniques to predict limit cycles for a wheelset
model with nonlinear effects, including coulomb friction. Their objective is to apply these techniques to a railway vehicle model to study unforced and forced response to track irregularities.

There exist clearances and stops in a freight truck between the wheelsets and sideframes and between the bolster and sideframes. If small motions are assumed, stops which limit large amplitudes of motion may be neglected. For example, warp and yaw motions of the truck frame are limited by the bolster and the wheel flanges, respectively. For small motions, these stops may be neglected. For nonlinear analyses, clearances and stops can be modeled as stiff deadband springs as shown in the sketch below.

If the actual force versus deflection and force versus velocity characteristics are known at the various connections throughout the truck, values of effective stiffness and damping which account for the clearances and stops can be determined by linearization techniques for varying amplitudes of motion.

For models of a freight car with roller bearing trucks, the clearances and compliance between the wheelsets and sideframes
bearing trucks were neglected and "ball-joint" connections were assumed by Blader and Kurtz [4] and by Law and Cooperrider (for the Freight Car Dynamics program). Blader's and Kurtz's model has been partially validated by experiment, and field tests were conducted in late 1976 and early 1977 to provide data for validation of the models developed in the "Freight Car Dynamics" project.

In plain bearing trucks, the lateral motion of the wheelsets with respect to the truck frame is opposed by sliding friction in the bearings and is limited at large amplitudes of lateral motion by stops in the journal box. Friction in the plain bearing/wheelset connections may be approximated by equivalent viscous damping elements. There is no restoring force acting at these connections, so that there may be effectively zero lateral primary stiffness provided by the plain bearings.

A consideration which influences the range of validity of a linear freight truck model is how the linearized secondary suspension groups are modeled. The secondary spring groups can be modeled with varying complexity and accuracy, as determined by the assumptions made for the spring height, the spring end conditions and degrees-of-freedom of the truck. The purpose of the model determines the sophistication of the spring group model. To study trends or stability of a generic vehicle model, a simpler spring group model may be adequate. A more complex and accurate spring group model may be required to study the dynamic behavior of a specific freight car design.

As previously described, the secondary spring groups rest in the sideframes and support the bolster at its ends. With this configuration, the springs ends are effectively fixed. Thus, they
can resist relative vertical, lateral and rotational motions between the sideframes and bolster. Further, the motions of the bolster and sideframes influence how the suspension forces and moments enter the equations of motion. Thus, the assumptions made for rigid body motions of the truck must be compatible with the spring model.

To study general trends in dynamic behavior, it is probably adequate to assume that the secondary spring groups have zero height and "guided" ends so that the springs have lateral and vertical components which act independently. This spring model is illustrated in the sketch shown below for the case of a two-dimensional slab on springs.

![Simple Spring Model.](image)

This type of spring model neglects the rotational moment exerted by the fixed ends. Tests conducted by American Steel Foundries [29] have shown that the resistance to relative roll motions across the springs that is provided by the fixed spring ends is small compared with the large contributions to the moment from the lateral and vertical stiffnesses. Thus, it may be reasonable to neglect the pure rotational spring moment when modeling the secondary suspension of a freight truck.
Spring height enters into the car body roll equation and may have a significant effect on roll motions. Thus, for "realistic" vehicle models, it is probably desirable to include the height of the secondary springs. For simplicity, the eleven degree-of-freedom (11 DOF) model developed for this report includes the secondary spring group model with zero height and guided ends just described. This is justified since the model is of a "pseudo" rail vehicle and was used to investigate general trends in the stability of different generic truck designs.

A more advanced (and accurate) secondary spring group model may consider a non-zero spring height and fixed end conditions of the springs. This spring model is illustrated in the sketch shown below for a two-dimensional slab on springs.

```
fixed ends

"Advanced" Spring Model.
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The forces and moments exerted at the spring ends may be assumed to have the form:

\[ Q_i = \sum_{j=1}^{3} k_{ij} q_j \]

where \( q_1 \), \( q_2 \), and \( q_3 \) are the lateral, vertical, and rotational displacements, respectively, across the spring group.

This model of a secondary spring group is included in the complete vehicle model described in Chapter 5. We are not
aware of experimental data from which the "off-diagonal" terms \((i \neq j)\),
can be defined, and these terms are neglected in the model.

In the 11 DOF model and in the complete vehicle model, roll
motions of the sideframes are neglected for simplicity. This
assumption is probably valid for worn sideframe/bearing adapter
connections. For new connections, there probably exists some roll
motion of the sideframe. Observations of sideframe roll motions
have been reported in informal conversations with various persons
connected with the railroad industry.

The models developed for this thesis consider the primary
and secondary suspension as linear spring and damper combinations.
Values for the suspension parameters are obtained from experimental
data or judiciously guessed. Conventional and unconventional
freight truck configurations are modeled by varying these parameters.
Suspension modeling of particular truck designs is discussed further
in Chapter 4.

Car Body/Bolster Connection

For conventional freight cars, the car body is supported at
the centers of the bolsters through centerplate connections.
The centerplate connection consists of a stub that is attached
to the car body and fits loosely in a shallow dish that is fixed
to the center of the bolster. Sometimes a "kingpin" is placed
through holes in the centers of the stub and dish.

The car body may rock in the centerplate with little restraint
from either the kingpin or the rim of the disk. Because of the
small clearance allowed, the car body may also slide a small amount
in the disk until the stub contacts the rim of the disk. In a severe
rocking case, the car body may lift completely off the bolster and roll until it contacts side bearings at the ends of the bolster. If high enough forces exist, the car body can roll up onto a sidebearing and completely off the trucks. This rocking motion usually occurs at certain forward speeds due to excitation from half-staggered rail joints. At other speeds, the car body will roll with the bolster.

Car body rocking is a nonlinear phenomenon, as friction exists at the centerplate connection and there are clearances between the centerplate stub and disk and between the car body and sidebearings. The nonlinear model may be solved by direct integration of the equations of motion. Alternatively, techniques can be used to linearize the nonlinear model to examine stability of the linear model.

Tse [30] developed a nonlinear freight car rocking model and found that large amplitudes of car body rocking occurred at speeds between about 10 to 20 mph, and diminished at higher speeds. Emerson [31] developed a two degree-of-freedom nonlinear rocking model and studied the response of the model to forcing from half-staggered rail joints. He observed large amplitude rocking motions, similar to those observed by Tse, for a speed range of about 15 to 20 mph. The model exhibited nonlinear "soft spring" and "jump" characteristics. As speed, and thus, forcing frequency, is decreased from above the range where large amplitude rocking occurs, the amplitude of rocking increases rapidly. As the speed is varied slightly in the range where large rocking motion begins
the amplitude of response can change suddenly. The "soft spring" effect and "jump" phenomenon are illustrated by the sketch shown below.

Typical Nonlinear Frequency Response Characteristics.

It is assumed for simplicity for this study that the car body rolls and moves laterally with the bolster. Results obtained to date have not clearly established the influence of car body rocking on stability. A rocking model is valuable for studying forced response of a vehicle over certain frequency ranges and is obviously needed to study severe rocking cases. Thus, the simplifying assumption may or may not be reasonable in all cases for predicting stability, but is valid for small motions of the car body in-phase with the bolster. Future stability studies should include a car body rocking model.

Vehicle Component Flexibility

If a structural member of a rail vehicle is sufficiently flexible, coupling of its flexural modes with rigid body modes can have a significant influence on the stability of the vehicle. It is just as important in stability studies to model the flexibility
of these members as it is important to model accurately other aspects (e.g. vehicle suspension, wheel/rail forces, etc.) of the actual vehicle. The structural members of a conventional rail vehicles are the wheelsets, sideframes, bolster and car body. The bolster and sideframes usually have fundamental flexural modes with frequencies which are high compared with the frequencies of the rigid body modes. The torsional modes of the wheelsets and the flexural and torsional modes of certain car bodies may be strongly coupled with the rigid body modes.

**Wheelset Torsional Flexibility**

A conventional wheelset consists of two wheels fitted tightly onto an axle so that the wheels move with the axle. Doyle and Prause [32] used a complete vehicle model to study the effect of axle torsional flexibility on stability. They found that the flexibility of conventional axles had a small effect on the critical speed. They also found that increasing the axle torsional flexibility decreased the critical speed of their model. Their study was completed at the same time as this investigation was being conducted.

Nayak and Tanner [33] theorized that torsional oscillations of conventional wheelsets may be the cause of the decrease in adhesion between the wheels and rails at high speeds. Decreases in available adhesion becomes a problem for adequate traction and braking particularly for locomotives. A solution to this problem has been offered by SAB [34], which has introduced a resilient wheel design that is used on some locomotives in England
and Western Europe. Elastomer pads are used to isolate the wheel tire from the hub so that the tires have torsional and lateral degrees-of-freedom with respect to the hub. The wheels are designed with the objectives of reducing noise transmission to the car from the track, increasing the available torque to the wheels, and decreasing wear and tear of the wheels and rails. Doyle and Prause [32] note that the flexible axle model is also a valid model of the torsional degree-of-freedom for a wheelset with resilient wheels.

The case for independently rotating wheels has been investigated by Kaplan and Short [35]. They considered independently rotating wheels with unconventional profiles on a wheelset model and a complete vehicle model. The profiles were such that the contact angles between the wheels and rails were zero. The found that this configuration would eliminate the truck hunting mode and introduce a stable, lightly damped mode. The also found that, by increasing the primary yaw stiffness, the guidance function of the wheelsets could be achieved.

Thus, the influence on stability of wheelset torsional flexibility warrants investigation. A parametric study of the effects of axle torsional stiffness on stability is presented in Chapter 3.

Car Body Flexibility

In many stability analyses of rail vehicles, the car body is assumed to be a rigid body. This assumption may be reasonable for certain car bodies that have flexural modes with natural frequencies that are high compared to the frequencies of the
rigid body modes. The assumption of a rigid car body, however, can be inaccurate for certain car body types.

There are about as many types of car bodies as there are cargoes that are carried by rail, and they have varying flexural properties. For example, hopper cars (which carry loose bulk materials) and box cars (which carry solid bulk material) have fundamental bending and torsional modes of lower frequencies than those of cylindrical tank cars (which are liquid carriers). Flat cars (which can carry large containeers, automobiles, etc.) may be more flexible in torsion and stiffer in lateral bending than box, hopper or tank cars. The type and amount of loading of a car body will also affect its flexural properties.

In Chapter 5, a study of the effects of car body flexibility on stability is presented. The nominal car body for this study is an 80 ton open hopper car. A flat car body is also considered.

Modeling Component Flexibility

A continuous element is an infinite degree-of-freedom system having an infinite number of oscillatory modes. An exact solution to the eigenvalue problem for the motion of a continuous system may not exist and usually is not feasible. Approximate methods can be used, however, to discretize the continuous element, i.e., model the element as a finite degree-of-freedom system. The number of degrees-of-freedom of the discrete model corresponds to the number of modes of oscillation that can exist. The accuracy of the discrete model increases with the number of degrees-of-freedom of the model, provided all model parameters are known and are accurate.
The two general types of approximation methods which discretize a continuous element are finite series solutions, such as the assumed modes method, and lumped parameter methods such as the method of influence coefficients [36]. The first type of approximation method assumes the motion of the system to be represented by a finite series in the form of the product of space-dependent functions and time-dependent generalized coordinates. The lumped parameter method lumps the mass of the element into discrete rigid elements, and lumps the flexibility of the element into suspension elements which connect the discrete masses.

Another approximation method for continuous elements is the finite element method. This method was developed for use on digital computers. It essentially considers a continuous element or complex structure as a system of discrete continuous elements connected at boundaries. At the boundaries, internal forces and displacements are made compatible so that the system behaves as one element or structure.

The car body model that is discussed in Chapter 5 is developed using a lumped parameter method. For simplicity, the car body is divided into two equal masses connected by torsional springs and dampers which act in the vertical and horizontal planes. In this manner, the first lateral and first torsional bending modes may exist.

The two-mass approximation of a car body can result in a "crude" model. By lumping the flexibility at one point, the influence
of the higher flexural and torsional modes on car body motions is neglected.

If the higher flexural and torsional modes have frequencies which are high compared to the fundamental frequencies and the rigid body frequencies, the two-mass car body model may be adequate for examining the stability of the vehicle model. Further, if the characteristics of the higher flexural modes are not determined and verified by experiment, a more elaborate car body model can be less accurate than the two-mass model.

Tests conducted by the Denver Division of Martin-Marietta Corporation [37] and tests conducted by ACF Industries Amcar Division and Pullman-Standard [38], indicate that, for the car bodies considered for this thesis, the fundamental frequencies of torsional and flexural bending may be low enough to affect significantly the motions of the vehicle which can go unstable. The nominal parameters which describe the flexibility of the hopper car and flat car models are obtained from results of these tests. Calculations of the effective stiffnesses and damping coefficients are discussed in detail in Reference [2].
CHAPTER 3. TORSIONALLY FLEXIBLE WHEELSET STUDY

Introduction

A study made using a three degree-of-freedom flexible wheelset model is described in this chapter. The model was developed as the first "block" in a "building block" approach to arrive at a complete vehicle model. A detailed derivation of the equations of motion for the model is given in Appendix A. Results and conclusions are discussed below.

The objectives of this study were 1) to determine the validity of assuming rigid wheelsets in stability studies of railway vehicles, 2) to investigate how the critical speed varies as a function of axle torsional stiffness, and 3) to investigate the effects of changes in other model parameters on the variation of critical speed with axle torsional stiffness.

Description of Model

The wheelset is considered to be perfectly symmetric and consists of two rigid wheels connected by a torsionally flexible axle. The flexibility of the axle is modeled as a linear torsional spring and damper in parallel. The wheelset is connected by suspension elements to a rigid truck that translates down the track at a constant speed, \( V \). The track is tangent and perfectly rigid so that the creep forces are determined by the motions of the wheelset only. Thus, the degrees-of-freedom of the model are lateral, yaw, and torsional displacements of the wheelset. Small displacement assumptions are made so that products of the perturbation quantities
may be neglected as second order terms. The effects of spin creep are included.

A schematic of the model is shown in Figure 3.1, and the nominal model parameters are listed in Table 3.1.

The flexible wheelset model was checked both analytically and numerically against an existing two-degree-of-freedom rigid wheelset model. By letting the axle torsional stiffness become infinitely large and the torsional displacement of the wheelset approach zero, the wheelset torsional equation of motion is identically zero, and the wheelset lateral and yaw equations of motion are identical to those for the rigid wheelset. By making the axle torsional stiffness very large, the eigenvalues and eigenvectors for the rigid and flexible wheelset models are extremely similar.

Equations of Motion

The lateral, yaw, and torsional motions of the wheelset are described by Equations (A.48), (A.49), and (A.50), respectively. Wheelset torsional motion is defined as the difference in angular displacements between the left and right wheels, $\beta_L - \beta_R$, about the $\hat{i}'$ axis (See Figure A.1 for definition of axis systems). As shown in Figure 3.1, $\beta_L$ and $\beta_R$ are both defined as about the positive $\hat{i}'$ axis, and $\beta_L = -\beta_R = \beta / 2$ from the symmetry of the first axle torsional mode. Then, $\dot{\beta}$ is defined as the net torsional spin about the positive $\hat{i}'$ axis, or twice the positive angular velocity of the left wheel.

The equation of motion for wheelset torsional motions shows the coupling of the lateral and torsional motions through the term $f_{33} \lambda_1 x$. This term may be defined physically as the moment generated about the wheelset axle by the components of the longitudinal creep forces
FIGURE 3.1 Schematic of Wheelset Model.


<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_w$</td>
<td>62.5 slugs (912.1 kg)</td>
</tr>
<tr>
<td>$I_{W1}$</td>
<td>100 slug ft$^2$ (135.5 kg m$^2$)</td>
</tr>
<tr>
<td>$I_{W2}$</td>
<td>348 slug ft$^2$ (471.5 kg m$^2$)</td>
</tr>
<tr>
<td>$a$</td>
<td>2.5 ft (0.762 in)</td>
</tr>
<tr>
<td>$r_0$</td>
<td>1.33 ft (0.405 m)</td>
</tr>
<tr>
<td>$\delta_0$</td>
<td>4.5 ft (1.372 m)</td>
</tr>
<tr>
<td>$d_0$</td>
<td>0.5 ft (0.152 m)</td>
</tr>
<tr>
<td>$k_{XP}$</td>
<td>$5.0 \times 10^4$ lb/ft ($7.297 \times 10^5$ N/m)</td>
</tr>
<tr>
<td>$k_{BP}$</td>
<td>$1.25 \times 10^6$ lb ft/rad ($1.69 \times 10^6$ N·m/rad)</td>
</tr>
<tr>
<td>$k_{AX}$</td>
<td>$2.26 \times 10^6$ lb ft/rad ($3.06 \times 10^6$ N·m/rad)</td>
</tr>
<tr>
<td>$V_c$</td>
<td>415 fps (126.5 m/sec)</td>
</tr>
<tr>
<td>$W_{AP}$</td>
<td>10,000 lb (44,480 N)</td>
</tr>
</tbody>
</table>
due to the difference in rolling velocity between the wheels. The terms which couple the torsional and yaw motions are $f_{33} \frac{r_0 a_b}{V}$ and $-g_r a_e$. The former term is the yaw moment generated about the wheelset centroid from the components of the longitudinal creep forces due to the different torsional displacements of the wheels. The latter term is the moment generated about the wheelset axle from those components of the longitudinal creep forces which arise from the yaw velocity of the wheelset. The equations show that torsional motions of the wheelset are independent of lateral and spin creep.

Results

Many parameters will affect the stability of a rail vehicle, and on tangent track, the most common stability indicator is the critical speed. The objective of the study described in this section was to investigate the effects of axle torsional flexibility on the critical speed of the wheelset model for various configurations. Results are illustrated by plots of the loci of eigenvalues in the complex plane and by "stability boundaries", which are curves of the speeds at which the real part of the eigenvalues changes sign for varying axle torsional stiffness (In the remaining discussions, the term "torsional stiffness" refers to "axle torsional stiffness").

Torsional stiffness is plotted as a function of axle diameter in Figure 3.2. It should be noted that resilient wheels (described in Chapter 2) also allow torsional motions of the wheelset. The results presented in this section refer to any wheelset configuration that permits torsional motions and that can be described by the equations of motion for this model.
FIGURE 3.2 Effect of Axle Diameter on Axle Torsional Stiffness
Modes of Oscillation

There exist two oscillatory modes for the wheelset model which can become unstable. One is a low frequency "wheelset hunting mode" (about 8 Hz at the high critical speeds) and the other an axle torsional or "spin mode", which is a relatively high frequency motion (about 15 Hz at critical speed for the nominal torsional stiffness).

The frequency, shape and stability of both modes are sensitive to the torsional stiffness. At a given forward speed, there exist ranges of torsional stiffness for which both oscillatory modes are unstable, and for which motions of the spin mode can go from stable to unstable or from unstable to stable. The instability of the wheelset model at low torsional stiffnesses is due to the higher frequency spin mode. At larger stiffnesses, the wheelset hunting mode is unstable while the spin mode is stable. These characteristics are illustrated in Figures 3.3 and 3.4. In these figures, the root-loci of eigenvalues for both modes are plotted for the nominal wheelset configuration for varying torsional stiffness and constant forward speed.

As mentioned previously, the shapes of the modes vary slightly with torsional stiffness. Generally the wheelset hunting mode consists of large lateral and yaw displacements, and relatively small torsional displacements. The yaw and torsional displacements are usually in phase and lead the lateral motion by about ninety degrees. The shape of the spin mode is typically like the wheelset hunting mode with large torsional motions. As shown in the discussion below, the phase difference between the torsional displacements and lateral and yaw displacements varies with torsional stiffness.
FIGURE 3.3 Variation of Eigenvalues of Wheelset Oscillatory Modes with Axle Torsional Stiffness for $V = 100$ mph. ($k_{AX} = 2.26 \times 10^6$ ft lb/rad)
FIGURE 3.4 Variation of Eigenvalues of Wheelset Oscillatory Modes with Axle Torsional Stiffness for $V = 300$ mph. ($k_{AX} = 2.26 \times 10^6$ ft lb/rad)
A root-locus plot of the eigenvalues for both modes for varying speed and for several values of torsional stiffness is shown in Figure 3.5. The locus for the spin mode is more sensitive to torsional stiffness than is the locus for the wheelset hunting mode. The direction of the locus for the spin mode with increasing speed changes in the clockwise direction for increasing torsional stiffness, while the direction of the locus for the wheelset hunting mode is not sensitive to torsional stiffness.

Referring to Figure 3.5, the speeds at which the real parts of the eigenvalues for the spin modes change sign, i.e., for which $\alpha = 0$, are:

- 138 mph for $k_{AX}/k_{AX_0} = 0.0015$; $\alpha$ changes from + to -
- 16 mph for $k_{AX}/k_{AX_0} = 0.070$; $\alpha$ changes from + to -
- 65 mph for $k_{AX}/k_{AX_0} = 0.088$; $\alpha$ changes from + to -
- 146 mph for $k_{AX}/k_{AX_0} = 0.088$; $\alpha$ changes from - to +.

For $k_{AX}/k_{AX_0} = 0.15$ and 1.0, the real part of the eigenvalues for the spin mode remains negative, indicating that the spin mode is always stable.

The "stability boundary" curves for the nominal wheelset configuration are shown in Figure 3.6. The speeds at which the real parts of the eigenvalues for each mode change sign are plotted against the nondimensional torsional stiffness (normalized by the torsional stiffness for a six inch diameter axle). The critical speed of the wheelset hunting mode approaches the critical speed for a rigid wheelset as the torsional stiffness increases. Numerically, the critical speed of the flexible wheelset for a nondimensional torsional stiffness of 10.0 is less than one percent greater than the critical speed for a two degree-of-freedom rigid wheelset model. This value of stiffness
Initial Points of Loci Correspond to $V = 25$ mph

**SPIN MODES**

1. $\frac{k_{AX}}{k_{AX_0}} = 0.0015$
2. $\frac{k_{AX}}{k_{AX_0}} = 0.070$
3. $\frac{k_{AX}}{k_{AX_0}} = 0.088$
4. $\frac{k_{AX}}{k_{AX_0}} = 0.150$
5. $\frac{k_{AX}}{k_{AX_0}} = 1.0$

$\frac{k_{AX}}{k_{AX_0}} = 2.26 \times 10^6$ lb ft/rad

**FIGURE 3.5** Variation of Eigenvalues of Wheelset Oscillatory Modes with Speed for Several Values of Axle Torsional Stiffness
Figure 3.6 Effect of Axle Torsional Stiffness on Stability of the Wheelset Model
corresponds to an axle diameter of about 10.7 inches. At the nominal torsional stiffness (corresponding to an axle diameter of six inches), the critical speed is about 1.7 percent higher than that for the rigid wheelset model.

At values of nondimensional torsional stiffness above 1.0, the wheelset hunting mode consists of large lateral and yaw displacements and torsional displacements that are less than ten percent of the yaw displacements. The torsional and yaw displacements lead the lateral displacements by about 90 degrees. In this range of torsional stiffnesses, the spin mode consists of large torsional displacements and smaller lateral and yaw displacements. The lateral displacements lag the torsional displacements, and the yaw displacements lead the torsional displacements by less than 90 degrees. These modes are sketched below in the complex plane. In these "phasor diagrams", the magnitudes of the vectors correspond to amplitudes of displacements, and the angles between the vectors correspond to the phase angles between the displacements of the wheelset. The vectors rotate in the complex plane as shown, with the angular velocity, \( \omega \), of the corresponding mode, and the magnitudes of the vectors grow or decay depending on the sign of the real part of the eigenvalue.

Wheelset Hunting Mode

Spin Mode

Wheelset Mode Shapes for \( \frac{k_{AX}}{k_{AX_0}} > 1.0 \)
At a nondimensional torsional stiffness of about 0.1, the critical speed of the wheelset hunting mode begins to increase rapidly with decreasing torsional stiffness. The critical speed for the spin mode increases rapidly with increasing stiffness at a value of nondimensional torsional stiffness of about 0.07. The stability boundary for the two modes intersect at a point of maximum critical speed for the configuration. Near this "cusp point", the mode shapes and frequencies of the two modes are very similar, so that there is strong coupling of hunting and torsional motions.

At the cusp point, the shape of both modes consists of extremely small torsional displacements and large lateral and yaw "hunting" motions, i.e., the yaw motion leads the lateral motion by about 90 degrees. The torsional displacements lag the lateral displacements by about 90 degrees for the spin mode, and lead the lateral displacement by about 120 degrees for the lower frequency wheelset hunting mode. The shape of the modes are sketched below in the complex plane.
For nondimensional torsional stiffnesses of about 0.01 to 0.08 for the nominal wheelset model, the spin mode becomes unstable at very low speeds. As the torsional stiffness approaches zero, the speed at which the spin mode becomes unstable approaches a constant value. At these very low torsional stiffnesses, the spin mode consists of large wheelset lateral, yaw, and torsional displacements with more torsion motion than yaw motion. The yaw displacements lead the lateral displacements by about 90 degrees. The torsional displacements lag the lateral displacements by about 30 degrees. At these low torsional stiffnesses, the wheelset hunting mode, which has a much higher critical speed than the spin mode in this range of $k_{AX}/k_{AX0}$, consists of large lateral, yaw and torsional displacements. The torsional displacements are about 60 percent of the yaw displacements. The torsional and yaw displacements lead the lateral displacements by about 90 degrees. These mode shapes are sketched below.

Wheelset Mode Shapes for $k_{AX}/k_{AX0} < 0.08$
As mentioned previously, the wheelset loses its directional capability when the axle torsional stiffness is zero. Mathematically, the model will then have a constant zero eigenvalue. This condition is sometimes called "neutrally stable". Physically, this means that the wheels rotate independently with different angular velocities such that, at a constant forward velocity, \( \omega_L = \omega_R \), where \( \omega_L \) and \( \omega_R \) are the angular velocities of the left and right wheels, respectively. If the wheelset is displaced a small amount, it will remain in the displaced configuration. As a result, rapid wheel and rail wear may occur because of a greater tendency for flange contact that accompanies the loss in guidance capability.

Effects of Design Parameters on Stability of Flexible Wheelset Model

Effect of Conicity

The stability boundary for varying torsional stiffness for wheel conicities of 0.0, 0.033, 0.05 (the nominal value), and 0.1 are plotted in Figure 3.7. For non-zero values of wheel conicity, the general shape of the stability boundary is the same. Wheel conicity is shown to have a large effect on the critical speed in the approximate range \( \frac{k_{AX}}{k_{AX_0}} > 0.1 \), and a smaller effect in the range \( 0.01 < \frac{k_{AX}}{k_{AX_0}} < 0.1 \). An increase in wheel conicity decreases the critical speed for all values of torsional stiffness. The destabilizing effect of wheel conicity is well known and is discussed in Chapter 2.

On tangent track, decreasing the wheel conicity maximizes the critical speed. For conventional wheels, however, decreasing the
FIGURE 3.7 Effect of Axle Torsional Stiffness on Stability of the Wheelset Model for Different Values of Wheel Conicity
wheel conicity reduces the guidance capability of the wheelset and reduces the stabilizing effect of the gravitational stiffness. Good curving generally requires high wheel conicity, which implies a trade-off between good curving performance and high critical speed on tangent track.

The stability boundary for zero conicity (i.e., cylindrical wheels) is shown in Figure 3.7. For torsional stiffnesses not in the approximate range $0.008 < \frac{k_{AX}}{k_{AX_0}} < 0.09$, the model is always "stable".

For zero conicity, there exists one, very lightly-damped mode of oscillation for the wheelset mode. There exist also four subsidences, of which two are larger (about 150 to 1000 rad/sec) and two are relatively small (about 0.5 to 10 rad/sec). The oscillatory mode is similar to the "spin" mode described earlier, and consists of large torsional displacements and smaller lateral and yaw displacements. For torsional stiffness in the range $\frac{k_{AX}}{k_{AX_0}} < 0.08$, the lateral displacements lead the torsional displacements by less than 90 degrees. For $\frac{k_{AX}}{k_{AX_0}} > 0.08$, the lateral displacements lead the torsional displacements by more than 90 degrees.

For zero conicity, the wheelset torsional and lateral equations of motion are uncoupled. The lateral and yaw and the yaw and torsional equations of motion remain coupled through the creep forces. The results shown in Figure 3.7 indicate that coupling of torsional motions of a sufficiently flexible and undamped axle and yaw motions can result in unstable oscillations for the wheelset model. These unstable (as opposed to a pure divergence) oscillations occur because the wheelset is attached to a reference "truck" by suspension elements.
The "truck" is constrained to translate down the track. Thus, even with cylindrical wheels, the wheelset can oscillate about the track centerline due to the restoring forces provided by the suspension.

For an actual rail vehicle, no self-centering action is provided, as the trucks and car body can move about the equilibrium position. The case for cylindrical wheels was prescribed for a more sophisticated single truck and car body (11 DOF) model (this model is described in Chapter 4). For this configuration, there exists a zero eigenvalue, which indicates a "neutrally stable" condition similar to the same condition for independently-rotating wheels. The zero eigenvalue indicates also the lack of self-centering action with cylindrical wheels.

It is probably impractical to use cylindrical wheels in practice. The critical speed is increased for the wheelset model because of the constraints placed on motions of the "truck". For the more realistic 11 DOF model, and for actual vehicles, there exists a condition of "neutral stability". For both models, the guidance capability provided by tapered wheels is lost and must be provided in another manner. Curving performance is also degraded with cylindrical wheels. There is no restoring force provided by the gravitational stiffness or lateral creep forces in the tread region. Thus, the tendency for gross sliding of the wheels on the rails is opposed only by the creep forces or wheel flanges. As a result, the vehicle would probably stay on the track because of the occurrence of flange contact only, and thus, would experience higher rates of flange wear.
Effect of Yaw Stiffness

The stability boundaries for varying torsional stiffness for primary yaw stiffnesses of one-half and twice the nominal yaw stiffness and for the nominal yaw stiffness are shown in Figure 3.8. The curves are effectively "translated" to the left when the yaw stiffness is decreased and to the right when the yaw stiffness is increased. The critical speed increases for nondimensional torsional stiffness greater than about 0.1 and less than about 0.05.

In the general range $0.01 < \frac{k_{AX}}{k_{AX_0}} < 0.1$, very large increases in critical speed may be achieved by adjusting the yaw stiffness and thus "translating" the curve either to the left or to the right. Outside of this general range, the critical speed increases significantly with increasing stiffness. This trend is well-known and consistent with results obtained with other models (see, for example [32] and [39]).

As with the wheel conicity, increasing the yaw stiffness will increase the critical speed on tangent track, but will have an adverse effect on the curving ability of the wheelset. Newland [21] shows that the ability of the truck to negotiate curves without slipping is a strong function of the yaw stiffness. He also shows that "optimum curving" occurred for zero yaw stiffness. For this case, the wheelsets are unconstrained and are free to align themselves with the radius of the curve. As a result, no slipping on the rails occurs. There probably exists some optimum yaw stiffness which will provide both good curving ability and a high critical speed.
FIGURE 3.8 Effect of Axle Torsional Stiffness on Stability of the Wheelset Model for Different Values of Primary Yaw Stiffness
Effect of Creep Coefficients

The nominal creep coefficients for the wheelset model (and for the other models developed in this report) were calculated using Kalker's linear theory. The stability boundaries for varying axle torsional stiffness for three sets of creep coefficients values are shown in Figure 3.9.

In general, changing the creep coefficients has a small effect on the critical speed for this model. For nondimensional torsional stiffnesses above about 0.08, the critical speed increases slightly for a decrease in creep coefficients from Kalker's values. Below this value, the trend is unclear.

Different results using different models have been obtained by others. For example, Doyle and Prause [32] used a passenger vehicle model with flexible wheelsets and found that an increase in creep coefficients caused a decrease in critical speed at low torsional stiffness and increase in critical speed at high torsional stiffnesses. Clark and Law [39] used a rigid truck model with rigid wheelsets and found that increasing the creep coefficients slightly decreased the critical speed. Thus, the effects of creep coefficients on critical speed are strongly influenced by the vehicle design.

As discussed in Chapter 2, linear creep theory implies that there is no limiting value of adhesion. Since the linearized creep forces act like viscous dampers, slip of the wheels on the rails cannot occur. However for small displacements, the linear creep model is a good approximation to actual creep phenomenon.
Figure 3.9 shows the effect of axle torsional stiffness on the stability of the wheelset model for different values of creep coefficients. The graph plots the speed (V, MPH) against the nondimensional axle torsional stiffness, $\frac{k_{AX}}{k_{AX_0}}$.

The plot includes lines for:
- 2 x nominal creep
- Nominal creep
- 1/2 x nominal creep

The nondimensional axle torsional stiffness, $\frac{k_{AX}}{k_{AX_0}}$, is given by $2.26 \times 10^6$ lb ft/rad.

The unstable region is indicated by arrows on the graph.
Effect of Torsional Damping

The stability boundaries for varying torsional stiffness for the nominal model configuration and for two different values for torsional damping coefficient are shown in Figure 3.10.

The addition of torsional damping increases the speed at which the spin mode goes unstable and has a small, practically negligible effect on the stability of the wheelset hunting mode. Also, the region of extremely low critical speed of the spin mode is eliminated.

A large increase in the amount of torsional damping may be achieved by fixing some type of rotary dampers to each wheel. Increased torsional damping may also be provided by resilient wheels of the type mentioned in Chapter 2, due to the elastomer material between the tire and hub. For a conventional wheelset, however, there is no noticeable increase in the critical speed due to torsional damping, and it appears that applying torsional dampers to a wheelset would be effective only for highly flexible axles or for highly resilient wheels.

We are aware of only one study in which the effects of axle torsional damping was considered. Kaplan and Short [35] considered torsional damping for a vehicle model with independently rotating wheels and unconventional wheel profiles. For their model, they found that the addition of torsional damping improved the ride quality by decreasing the lateral accelerations of the vehicle at low yaw stiffnesses. They did not consider the effects of torsional damping on stability.
FIGURE 3.10 Effect of Axle Torsional Stiffness on Stability of the Wheelset Model for Different Values of Axle Torsional Damping.
Conclusions

1. The torsional flexibility of a conventional (six inch) freight car axle has a small effect on the critical speed for this model. Thus, the torsional flexibility of conventional axles can probably be neglected in stability analyses of freight cars with good accuracy.

2. There exists a maximum value for critical speed which occurs at the intersection of the "stability boundary curves" for the spin and wheelset hunting modes. This occurs for a torsional stiffness corresponding to an axle diameter of about three inches, and the critical speed is increased by about 25 percent over the critical speed for the nominal wheelset model (with a six inch diameter axle).

A three inch axle diameter may be unrealistic from the point of view of load-carrying capacity and fatigue life of the axle. Thus, an unconventional wheelset design may be necessary to provide this "optimum" torsional stiffness. One possibility would be to use resilient wheels on conventional axles. For conventional wheels, the range in axle diameters found on North American rail vehicles is about 4.25 to 7.38 inches [40]. This range of diameters corresponds to $0.25 < \frac{k_{AX}}{k_{AX_0}} < 2.28$.

3. For the range of torsional stiffnesses mentioned in Conclusion 2, the critical speed of the wheelset model is affected to a very small degree by a change in creep coefficient and by the addition of torsional damping. The influence of these parameters on the stability of a freight car is questionable, as the wheelset model used in this study considers the dynamics of a single wheelset only.
The coupling of wheelset motions with other vehicle motions can produce significantly different results. However, a simple model of this type is valuable for providing insight for interpreting behavior of more complex models.

4. Increases in the yaw stiffness and decreases in wheel conicity will increase the critical speed of the model. These effects are well known and have been shown with other models. Trade-offs exist, however, between good curve negotiation and high critical speed.

5. For conventional wheelsets and track, the implementation of independently rotating wheels and/or cylindrical wheels appears impractical. For both cases, the guidance function of the wheelset is lost. Also, curving performance is degraded and wheels and rails may wear more rapidly because of a greater tendency for flange contact. The critical speed of the wheelset model is increased for cylindrical wheels only because the constraints placed on motions of the reference "truck" result in restoring forces which center the wheelset. For actual vehicles, there is no self-centering action.
CHAPTER 4. SUSPENSION STUDY

Introduction

A study made using an eleven degree-of-freedom (11 DOF) model of a single truck and "pseudo" car body is described in this Chapter. The effects of the addition of a primary suspension to a freight truck, and variations in the primary suspension and the truck warp stiffness on the stability of various configurations was investigated.

The objectives of this study were 1) to investigate the possible benefits of adding a primary suspension to a freight truck, 2) to develop a general truck model with which various types of trucks can be modeled by simply changing some design parameters, 3) to compare the stability of the varous types of trucks, 4) to examine the adequacy of the modeling of a North American roller bearing freight truck as having "ball joint" wheelset/sideframe connections and 5) to investigate the possible benefits of using interconnected wheelsets on a freight truck.

Description of Model

The truck consists of two wheelsets connected to the truck frame by lateral and longitudinal suspension elements. The truck frame consists of two sideframes and a bolster connected by lateral and vertical suspension elements. Each wheelset has lateral, yaw, and torsional degrees-of-freedom. The two sideframes are assumed to move together laterally and in yaw. As mentioned in Chapter 2, sideframe roll motions have been neglected. In addition, the side-
frames can "warp", or rotate about the truck centerline to "form a parallelogram" in the horizontal plane. The bolster moves laterally with respect to the sideframes, and yaws about its center of gravity such that it remains parallel to the lines through the ends of the sideframes.

The "pseudo" car body consists of a rigid mass which rolls in the vertical plane and moves laterally with the bolster. The bolster is assumed to roll with the car body.

Thus, the eleven degrees-of-freedom for the model are: lateral, yaw, and spin motions of each wheelset; lateral, yaw, and warp motions of the truck frame; and lateral and roll motions of the car body. Schematics of the 11 DOF model are shown in Figures 4.1 to 4.3.

By prescribing limiting cases, different types of trucks are obtained. For example, by letting the "warp" stiffness of the truck frame become very large, the truck model becomes a passenger truck model. Then, by making the primary suspension very stiff on the passenger truck model, a rigid truck model is obtained.

A roller bearing truck model is obtained by making the primary suspension very stiff at the nominal value of warp stiffness. By making the axle torsional stiffness large, the roller bearing truck model approaches the five degree-of-freedom (5 DOF) model previously developed in the Freight Car Dynamics program conducted by Clemson University and Arizona State University.

A plain bearing truck model is obtained by increasing the longitudinal primary stiffness, setting the lateral primary suspension stiffness equal to zero, and approximating sliding friction in the
FIGURE 4.1 Schematic of 11 DOF Model-Equilibrium Configuration (Plan View).
FIGURE 4.2 Schematic of 11 DOF Model-Disturbed Configuration (Plan View).
a) Equilibrium Configuration. b) Disturbed Configuration.

FIGURE 4.3 Schematic of 11 DOF Model (Front View).
journal box by an equivalent viscous damping coefficient (equivalent viscous damping approximations are made for the friction which exists throughout an actual vehicle).

Various wheelset interconnection configurations are obtained by adjusting the bending and shear stiffnesses between the wheelsets. Other new and unconventional truck designs can be obtained by adjusting the suspension parameters in different combinations.

Equations of Motion

A detailed derivation of the equations of motion is presented for this model in Reference [2]. The equations are given in matrix form in Appendix B.

To arrive at the three truck equations, the equations of motion for each sideframe and the bolster are derived. Then, combinations of these equations will describe the lateral, yaw, and warp motions of the truck. Summing the two sideframe lateral equations yield the truck lateral equation. The truck yaw equation is obtained by adding the bolster yaw equations, the two sideframe yaw equations, and the two sideframe longitudinal equations times half the distance between the sideframes. The truck warp equation is obtained by adding the bolster yaw equation and the two sideframe longitudinal equations times half the distance between the sideframes.

The front and rear wheelset equations are essentially the same as those derived in Appendix A. Changes are made to include the longitudinal and lateral primary suspension elements. The wheelset lateral and yaw displacements are defined relative to the instantaneous
truck centerline. This was done so that the mode shapes of the model could be more easily identified from the eigenvectors.

Results

A two-part study was conducted with the 11 DOF model. The effects on stability of adding primary suspension and of varying the warp stiffness and axle torsional stiffness on a freight truck were examined in the first part. In the second part, the effects on stability of variations in bending and shear stiffnesses between wheelsets were examined for a generic truck model with wheelset interconnection.

Freight Truck Study

The nominal model configuration for this study is a modified 70 ton ASF Ride Control truck and "pseudo" car body representative of a light L & N 80 ton open hopper car. The truck is modified by the addition of primary suspension springs with stiffnesses corresponding approximately to those on the LIMRV vehicle [14]. The nominal model parameters are listed in Table 4.1.

As previously discussed, the 11 DOF model was "validated" by comparing its behavior in a limiting case against the 5 DOF model. The primary suspension stiffness of the 11 DOF model was increased in an attempt to "converge" to the 5 DOF model with "ball joint" wheelset/sideframe connections. The results of this "validation" process are shown in Figure 4.4, where critical speed is plotted against nondimensional primary lateral stiffness for a constant ratio $k_x/k_z$. For primary lateral and longitudinal stiffnesses that...
TABLE 4.1

NOMINAL DESIGN PARAMETERS FOR ELEVEN DEGREE-OF-FREEDOM (11 DOF) MODEL

<table>
<thead>
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<th></th>
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</thead>
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<td>( m_w ) = 76.6 slugs</td>
<td>( m_B ) = 36.1 slugs</td>
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<td>(526.8 kg)</td>
</tr>
<tr>
<td>( I_{w1} ) = 53.1 slug ft(^2)</td>
<td>( I_{B2} ) = 178.6 slug ft(^2)</td>
</tr>
<tr>
<td>(72.0 kg m(^2))</td>
<td>(242.1 kg m(^2))</td>
</tr>
<tr>
<td>( I_{w2} ) = 448.5 slug ft(^2)</td>
<td>( I_{B3} ) = 178.6 slug ft(^2)</td>
</tr>
<tr>
<td>(608.1 kg m(^2))</td>
<td>(242.1 kg m(^2))</td>
</tr>
<tr>
<td>( m_s ) = 24.0 slugs</td>
<td>( m_c ) = 551.0 slugs</td>
</tr>
<tr>
<td>(350.2 kg)</td>
<td>(8041.2 kg)</td>
</tr>
<tr>
<td>( I_{s2} ) = 77.6 slug ft(^2)</td>
<td>( I_{c3} ) = 6500.0 slug ft(^2)</td>
</tr>
<tr>
<td>(105.2 kg m(^2))</td>
<td>(8812.8 kg m(^2))</td>
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</table>

<table>
<thead>
<tr>
<th>Geometry</th>
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<tbody>
<tr>
<td>( \lambda ) = 2.8 ft</td>
<td>( d ) = 3.25 ft</td>
</tr>
<tr>
<td>( r_0 ) = 1.375 ft</td>
<td>(0.991 m)</td>
</tr>
<tr>
<td>( a ) = 2.46 ft</td>
<td>( h ) = 2.994 ft</td>
</tr>
<tr>
<td>( a ) = 2.46 ft</td>
<td>(0.750 m)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Wheel/Rail Contact Parameters*</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{11} ) = 994,670 lb/wheel</td>
<td>( \Delta_1 ) = 0.0</td>
</tr>
<tr>
<td>( (4.426 \times 10^6 \text{ N/wheel}) )</td>
<td>( \delta_0 ) = 0.05</td>
</tr>
<tr>
<td>( f_{12} ) = 4607 lb ft/wheel</td>
<td>( a_1 ) = 0.05</td>
</tr>
<tr>
<td>( (6245.3 \text{ N-m/wheel}) )</td>
<td>( \lambda_1 ) = 0.05</td>
</tr>
<tr>
<td>( f_{22} ) = 77.5 lb ft(^2)/wheel</td>
<td>( \lambda_1 ) = 0.05</td>
</tr>
<tr>
<td>( (32.02 \text{ N-m}/\text{wheel}) )</td>
<td></td>
</tr>
<tr>
<td>( f_{33} ) = 1.153 ( \times 10^6 ) lb/wheel</td>
<td></td>
</tr>
<tr>
<td>( (5.129 \times 10^6 \text{ N/wheel}) )</td>
<td></td>
</tr>
</tbody>
</table>

*Creep coefficients are obtained from Kalker's linear theory.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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</thead>
<tbody>
<tr>
<td>$-k_x^p$</td>
<td>$2.9 \times 10^5$ lb/ft</td>
<td>(4.23 $\times 10^6$ N/m)</td>
</tr>
<tr>
<td>$k_z^p$</td>
<td>$3.8 \times 10^5$ lb/ft</td>
<td>(5.55 $\times 10^6$ N/m)</td>
</tr>
<tr>
<td>$k_{AX}$</td>
<td>$2.26 \times 10^6$ lb ft/rad</td>
<td>(3.06 $\times 10^6$ N·m/rad)</td>
</tr>
<tr>
<td>$k_x$</td>
<td>24,000 lb/ft</td>
<td>(350,000 N/m)</td>
</tr>
<tr>
<td>$k_y$</td>
<td>$2.658 \times 10^5$ lb/ft</td>
<td>(3.879 $\times 10^6$ N/m)</td>
</tr>
<tr>
<td>$k_{\theta CP}$</td>
<td>50.0 lb ft/rad</td>
<td>(67.6 N·m/rad)</td>
</tr>
<tr>
<td>$k_{\theta W}$</td>
<td>$4.011 \times 10^6$ lb ft/rad</td>
<td>(5.42 $\times 10^6$ N·m/rad)</td>
</tr>
<tr>
<td>$k_x^L$</td>
<td>0.0 lb/ft</td>
<td>(0.0 N/m)</td>
</tr>
<tr>
<td>$k_y^L$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_{APP}$</td>
<td>12,695 lb</td>
<td>(56,467.4 N)</td>
</tr>
<tr>
<td>$V_c$</td>
<td>136.7 fps</td>
<td>(41.5 m/sec)</td>
</tr>
</tbody>
</table>

$D_x^p = 0.0$ lb sec/ft (0.0 N·sec/m)

$D_{z^p} = 0.0$ lb sec/ft (0.0 N·sec/m)

$D_{AX} = 0.0$ ft·lb·sec/rad (0.0 N·m·sec/rad)

$D_{X} = 1.2645 \times 10^4$ lb sec/ft (1.8441 $\times 10^5$ N·sec/m)

$D_y = 7587$ lb sec/ft (110,600 N·sec/m)

$D_{\theta CP} = 2409$ ft·lb·sec/rad (3268 N·m·sec/rad)

$D_{\theta W} = 9318$ ft·lb·sec/rad (12,640 N·m·sec/rad)

$W_{APP} = 12,695$ lb (56,467.4 N)
FIGURE 4.4 Effect of Primary Suspension Stiffness on Critical Speed of the 11 DOF Model (Nominal Warp Stiffness)
are 100 times the nominal (LIMRV) values, the critical speed of the 11 DOF model is within one percent of that of the 5 DOF model. These high values of stiffnesses are of the order of those measured in tests conducted by Brenco [41] for roller bearing connections. The frequency, damping, and mode shape of the least-damped or "truck hunting" modes for the 5 DOF and "stiff" 11 DOF models are very similar. These results indicate that modeling the wheelset/sideframe connections as "ball joints" is adequate for stability analyses.

The critical speed of the model for the nominal primary suspension stiffness is about 8.5 percent higher than the critical speed for the high primary suspension stiffness, or "roller bearing truck" configuration. At one-half the nominal values of primary suspension stiffness, the critical speed is about 15.5 percent higher than for the roller bearing truck configuration. Below these values, the critical speed decreases rapidly. At one tenth the nominal values or primary suspensions stiffness, the critical speed is about 50 percent lower than that for the roller bearing truck configuration. Increases in critical speed from the roller bearing truck configuration of about 5 to 16 percent are achieved by adding primary suspension with stiffnesses corresponding approximately to from 40 to 120 percent of the nominal (LIMRV) values. The nominal value of warp stiffness was used for the model.

In an attempt to model a plain bearing truck, the longitudinal primary suspension stiffness was made "stiff" by increasing it to 100 times the nominal value. The lateral primary suspension stiffness
was set equal to zero, and the lateral primary damping coefficient was varied. The effect of lateral primary damping on critical speed for this configuration is shown in Figure 4.5.

As mentioned in Chapter 2, the critical speed of freight cars with plain bearing trucks has been observed to be about 10 mph lower than for freight cars with roller bearing trucks. For a lateral primary damping coefficient of about $4.5 \times 10^4$ lb-sec/ft, the critical speed of the 11 DOF model is about 10 mph less than the critical speed for the roller bearing truck configuration. This value of primary lateral damping seems rather high. Assumptions that seem reasonable for calculating an equivalent viscous damping for friction in the plain bearings are: 1) the amplitude of wheelset lateral displacement relative to the journal box (zero to peak) is 0.5", 2) the frequency of oscillation near the critical speed is 1.5 Hz, 3) the coefficient of friction between the axle and journal box is 0.16 (corresponding to steel sliding on lubricated steel), and 4) the normal load at each journal box is equal to one half of the axle load. The calculated equivalent viscous damping using these assumptions is about $3.3 \times 10^3$ lb sec/ft, which is about 90 percent less than the value which gives a critical speed thought to be representative of a plain bearing truck.

From Figure 4.6, possible combinations of values for amplitude of lateral motion, frequency, and coefficient of friction may be obtained to yield the "plain bearing truck" value of lateral primary damping coefficient ($4.5 \times 10^4 \frac{1\text{b-sec}}{\text{ft}}$) for $k_p = 0$. To obtain this value, either the frequency or amplitude of motion must be assumed
FIGURE 4.5 Effect of Primary Lateral Damping on Critical Speed of the 11 DOF Model $\frac{k_{x_p}}{k_{x_p_0}} = 0$, $\frac{k_{z_p}}{k_{z_p_0}} = 100$, Nominal Warp Stiffness).
FIGURE 4.6 Frequency vs. Wheelset Lateral Displacement for Different Coefficients of Friction: Plain Bearing Truck Configuration ($k_e = 0.0$, $D_e = 4.5 \times 10^4$ lb sec/ft).
extremely low (about 0.07 Hz or 0.003 inches, respectively), or the coefficient of friction assumed extremely high (about 1.0). The low value of lateral primary damping coefficient obtained from the equivalent viscous damping approximation using the four above-mentioned "reasonable" assumptions ($D_{xp} = 3.3 \times 10^3$ lb-sec/ft) was used, and lateral primary stiffness was added to check the effect on the critical speed of the model. Since the only lateral restoring force existing for a plain bearing configuration is provided at large lateral wheelset displacements by stops, the effective primary stiffness provided by the plain bearing is, at best, very small for small displacements. A critical speed representative of a plain bearing truck was reached for values of nondimensional lateral primary stiffness greater than about 0.2. These values of lateral primary stiffness may be too stiff for small amplitudes of wheelset lateral motions.

Large sustained limit cycle oscillations of the wheelsets, where lateral displacements are limited by the stops in the journal boxes, may result in a large effective primary lateral stiffness. For this case, the journal box/wheelset connections may be characterized by a very stiff deadband spring. Assuming that the ratio of the amplitude of lateral wheelset displacement (relative to the journal boxes) to the available clearance in the journal boxes is about 1.015, the describing function for the deadband spring gives an effective stiffness that is roughly 0.2 percent of the stiffness of the stops [42]. Assuming that the stiffness of a stop is of the order of the lateral stiffness of a roller bearing connection (about $2.9 \times 10^7$ lb/ft), then the describing function gives an effective value of lateral primary stiffness that is about twenty percent of
the nominal (LIMRV) stiffness. Assuming the clearance between the wheelset and stop is 0.5 inches, a value of nondimensional primary lateral stiffness of 0.2 implies that the maximum deflection of the stop during contact with the wheelset is about 0.008 inches.

A stability analysis addresses the question of the stability of motion following a small disturbance from equilibrium. The model of the later wheelset/sidefram connection, i.e. zero stiffness, used for the plain bearing truck is therefore reasonable for a stability analysis, as "small" motions would not result in contact of the axle with the stops. The motion observed for the vehicle with worn plain bearing trucks in the Seaboard Coast Line/Pullman Standard Tests [43] was a limit cycle oscillation which, in all likelihood, entailed contact of the axle with the stops. This, the mode shape obtained with the high damping value \(4.5 \times 10^4\) lb-sec/ft and zero lateral stiffness should not necessarily be expected to match that observed in tests while the vehicle was in a limit cycle condition. However, if a value of stiffness of 20 percent of the nominal (LIMRV) value is used with the "reasonable" damping value \(3.3 \times 10^3\) lb-sec/ft, the mode shape obtained is similar to that observed in the Seaboard Coast Line/Pullman-Standard tests.

Thus, it is questionable whether contact with the stops (i.e., limit cycle motions) should be modeled in stability analyses. This would mean using the "reasonable" value of lateral damping and non-zero lateral stiffness. If contact with the stops is not modeled, and relative motions are "small", there is no restoring force provided, and the lateral primary stiffness should be zero. Thus, the "high"
value of damping would be appropriate to use. For this study, the latter model is chosen as the "plain bearing truck" model (i.e., \( k_x = 0, D_x = 4.5 \times 10^4 \text{ lb-sec/ft} \)). The value of damping may be consistent with very small wheelset displacements relative to the journal box (see Figure 4.6).

A linear iterative technique has been used by Cooperrider et al [28] to find the amplitude and frequency of limit cycles and the speeds at which they occur. This technique could be employed to predict the limit cycle oscillations for plain bearing configurations.

The effect of warp stiffness on critical speed of the plain bearing truck model is shown in Figure 4.7. The critical speed of the model decreases rapidly as the warp stiffness is decreased or increased from the nominal value. Thus, the critical speed for this model cannot be increased by modifying the warp stiffness. Similar results were obtained by Cooperrider and Law [44] for a nine degree-of-freedom (9 DOF) model of a freight car with a roller bearing truck. The effect of lateral primary stiffness on critical speed for a "stiff" longitudinal primary suspension and nominal warp stiffness is shown in Figure 4.8. The critical speed is zero for lateral primary stiffness below about 1000 times less than the nominal stiffness. The critical speed increases with increasing lateral stiffness until the nominal (LIMRV) stiffness is reached. Then, as the lateral primary stiffness is increased above the nominal value, the critical speed decreases slightly and approaches the critical speed for the roller bearing truck configuration.
FIGURE 4.7 Effect of Warp Stiffness on Critical Speed of the 11 DOF Model for the Plain Bearing Truck Configuration.

NONDIMENSIONAL WARP STIFFNESS, \( \frac{k_{\theta W}}{k_{\theta W0}} \)

\( (k_{\theta W0} = 4.011 \times 10^6 \text{ lb ft/rad}) \)

CRITICAL SPEED, \( V_c \), MPH

0  2  4  6  8  10.0

0  20  40  60  80  100  120
FIGURE 4.8 Effect of Primary Lateral Stiffness on Critical Speed of the 11 DOF Model ($\frac{k_z p}{k_z p_0} = 100$, Nominal Warp Stiffness)

Nondimensional Primary Lateral Stiffness, $\frac{k_x}{k_{x p_0}}$

$k_x = 2.9 \times 10^5 \text{ lb/ft}$
The effect of lateral and longitudinal primary suspension stiffness on critical speed is shown in Figure 4.9 for a value of warp stiffness that is $10^4$ times higher than the nominal freight truck value. This value of warp stiffness makes the truck frame effectively rigid. For the nominal primary suspension, the critical speed is about 4.5 percent higher than for the nominal model. For values of primary suspension from ten percent of the nominal values to the full nominal values, the truck has characteristics typical of passenger trucks, i.e., a truck frame and soft primary suspension.

The critical speed decreases with increasing primary stiffness and is zero at "stiff" values of primary stiffness that are 50 times higher than the nominal (LIMRV) values. For a stiff primary suspension and a stiff truck frame, the truck is effectively rigid. The critical speed is zero due to the low secondary yaw stiffness for the model (50 lb ft/rad). As shown by Clark and Law [44], the secondary yaw stiffness is a stabilizing effect for a vehicle with rigid trucks. When the secondary yaw stiffness is made large, $(k_{\theta CP} = 2.666 \times 10^5$ lb ft/rad), the critical speed for the "rigid" critical speed for the "rigid truck" model increases from zero to about 30 mph.

There is very little yaw restoring force provided at the centerplate for conventional freight cars. Thus, a rigid truck design, with a conventional centerplate connection, is not desirable from the point of view of low, and even zero, critical speed, as well as poor curving performance due to the constraint placed on motions of the wheelsets relative to the truck frame.
FIGURE 4.9 Effect of Primary Suspension Stiffness on Critical Speed of 11 DOF Model.

\[
\frac{k_p}{k_p^0} = \begin{cases} 
2.9 \times 10^5 & (k_p = 2.9 \times 10^5 \text{ lb/ft}) \\
3.8 \times 10^5 & (k_p = 3.8 \times 10^5 \text{ lb/ft})
\end{cases}
\]
Thus, roller bearing truck, plain bearing truck, passenger truck, rigid truck configurations and a model of a freight truck with primary suspension elements have been obtained with the 11 DOF model by adjusting the primary suspension parameters and the warp stiffness. The "roller bearing truck (RBT)" model consists of a "stiff" primary suspension that has stiffnesses which are 100 times the nominal values, and nominal warp stiffness. The "plain bearing truck (PBT)" model consists of stiff longitudinal primary suspension (100 times the nominal value), zero lateral primary stiffness, a lateral primary damping coefficient of $4.5 \times 10^4$ lb sec/ft and nominal warp stiffness. The "passenger truck (PT)" model consists of the nominal primary suspension and a high warp stiffness ($10^4$ times the nominal value) which corresponds to a truck frame that is effectively rigid. The "nominal model" consists of a 70 ton ASF Ride Control truck frame but a primary suspension typical of the LIMRV vehicle. This may be thought of as a modified freight truck (MFT) model. The stability of these four models is now compared.

The frequency and damping ratio of the least-damped or "hunting" mode for the four models are shown in Figure 4.10. The frequency is almost linear with speed for each model. At a given speed, the frequency for the MFT model is slightly higher than the frequencies for other models. The frequency of the RBT model is the lowest of the four cases. The frequency of the PBT and PT models are about the same. The damping ratios for the nominal, RBT and PT models decrease with speed. The damping ratio for the PBT model increases with speed for speeds up to about 35 mph, and then decreases with speed.
FIGURE 4.10 Frequency and Damping Ratio vs. Speed for Different Truck Configurations of 11 DOF Model.
The critical speed for each model is the speed at which the damping ratio changes sign from positive to negative. As indicated by the damping ratio versus speed curves in Figure 4.10, the MFT model has the highest critical speed (93.2 mph), followed by the PT model (87.0 mph), the RBT model (83.2 mph) and the PBT model (76.4 mph). For a 50 percent decrease in the primary suspension stiffnesses of the PT model, the critical speed of the PT model is about 37 percent higher than the highest critical speed of the MFT model.

The mode shapes for the "hunting" modes of the four models are sketched in Table 4.2. For all models, wheelset torsional displacements are practically negligible. For the RBT model, the side-frames and wheelsets move effectively as a four-bar linkage. Car body and truck lateral displacements are about equal in magnitude, and angular displacements of the vehicle are about equal in magnitude. Car body lateral and roll displacements, and truck lateral displacements are nearly in phase. The truck yaw and truck warp displacements lead the truck lateral displacements by about 90 and 180 degrees, respectively.

The mode shape for the PBT model consists of large front and rear wheelset lateral displacements. Truck and car body lateral displacements are small by comparison. Angular displacements of the vehicle are about equal in magnitude and imply small angular motions of the vehicle. Relative to front wheelset lateral displacements, rear wheelset lateral displacements and truck lateral displacements are about 180 degrees out of phase, truck yaw displacements are nearly in phase and truck warp displacements lag by about 90 degrees.
<table>
<thead>
<tr>
<th>Truck Type</th>
<th>$V_c$ (mph)</th>
<th>$f_c$ (Hz)</th>
<th>Mode Shape (Plan View)</th>
<th>Mode Shape (Rear View)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roller Bearing Truck (RBT)</td>
<td>83.2</td>
<td>1.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Plain Bearing Truck (PBT)</td>
<td>76.4</td>
<td>1.49</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passenger Truck (PT)</td>
<td>87.0</td>
<td>1.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal Truck (MFT)</td>
<td>93.0</td>
<td>2.08</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Car body roll displacements are nearly in phase and car body lateral displacements lag the front wheelset lateral displacements by about 90 degrees.

The mode shape for the PT model consists of lateral displacements of the wheelsets, car body and truck that are about equal. Wheelset yaw displacements are the largest angular displacements, and truck yaw and car body roll displacements are small by comparison. Truck warp displacements are negligibly small. Truck, car body, and front wheelset lateral displacements are nearly in phase. Car body roll and truck yaw displacements lag the truck lateral displacements by about 90 degrees. Rear wheelset lateral, and front and rear wheelset yaw displacements are each about 180 degrees out of phase with respect to truck lateral displacements.

The mode shape for the MFT model consists of large car body and truck lateral displacements, and relatively small wheelset lateral and yaw, truck yaw and warp, and car body roll displacements. Truck lateral and car body lateral and roll displacements are nearly in phase. Truck yaw displacements lag, and truck warp displacements lead truck lateral displacements by about 90 degrees. Car body roll displacements are nearly in phase with truck lateral displacements. Front and rear wheelset lateral displacements lead the truck lateral displacements by about 120 degrees. Front wheelset yaw displacements lead the truck lateral displacements by about 130 degrees, and the rear wheelset yaw displacements lag the truck lateral displacements by about 75 degrees.

The effect of axle torsional stiffness on the critical speed of the nominal model is shown in Figure 4.11. The results differ
FIGURE 4.11 Effect of Axle Torsional Stiffness on Critical Speed of Nominal 11 DOF Model.
from those presented in Chapter 3 for the wheelset model, but are quite similar to the results obtained by Doyle and Prause [32] for their vehicle model. The critical speed decreases with decreasing axle torsional stiffness and is zero below an axle torsional stiffness which corresponds to an axle diameter of about 2.5 inches. The unstable mode is similar to the hunting mode for the MFT model but the amplitudes of the axle torsional displacements vary with the stiffness.

The critical speed of the wheelset model is non-zero for all values of axle torsional stiffness and, for certain low values of stiffness, the critical speed decreased with increasing axle torsional stiffness. The results shown in Figure 4.11 are more realistic than the trends discussed in Chapter 3, as they apply to a vehicle rather than an isolated wheelset. As discussed previously, the wheelset model does not account for the coupling of the truck and car body motions with motions of the wheelset. The 11 DOF model accounts partially for truck and car body motions, and should identify more accurately trends in the stability of an freight car.

Wheelset Interconnection Study

As mentioned previously, the 11 DOF model was modified to study the stability with interconnected wheelsets. A schematic of the wheelsets without the rest of the vehicle is shown in Figure 4.12.

The wheelsets are assumed to be connected to each other by a yoke or frame. The mass and inertia of the frame are lumped into the two wheelsets. For simplicity, the effect of the frame on the radius of gyration of the wheelset is neglected, and only the added
FIGURE 4.12 Schematic of Interconnected Wheelset Model (Plan View)
mass affects the moment of inertia of the wheelsets. As a result, the mass and yaw moment of inertia of each wheelset is increased by about 20 percent. The wheelset/sideframe connections are assumed to consist of elastomer pads between the sideframes and roller bearings. The pads provide more yaw compliance of the wheelsets with respect to the sideframes than could exist in a nominal roller bearing truck.

The model is intended to be generic and not represent any specific design. Some of the nominal design parameters are "ballpark" estimates of the "Prototype I" truck discussed by List et al [25, 46]. The nominal values of the parameters are listed in Table 4.3.

The study performed with this model examined the effect on critical speed of variations in the bending and shear stiffnesses, $k_\theta_L$ and $k_x_L$, respectively, between the wheelsets. The effect of shear stiffness on critical speed for three values of bending stiffness is shown in Figure 4.13. The critical speeds for the four truck configurations discussed in the previous section are plotted for comparison.

The critical speed of the vehicle with increased wheelset mass and yaw moment of inertia (due to the interconnection yoke or frame) but with zero values for $k_x_L$ and $k_\theta_L$ and reduced $k_z_p$ is about 70 mph. At a value of interconnection bending stiffness of $1.67 \times 10^6$ ft lb/rad, the critical speed exceeds 70 mph for all values of stiffness explored. At a higher value of bending stiffness ($3.34 \times 10^6$ ft lb/rad, the critical speed exceeds 70 mph up to a value of shear stiffness of about $7 \times 10^5$ lb/ft. Above this value of shear stiffness, the critical speed decreases to about 60 mph. For values
TABLE 4.3

CHANGES TO 11 DOF MODEL NOMINAL
DESIGN PARAMETERS FOR WHEELSET INTERCONNECTION

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_w$</td>
<td>92.13 slugs (1344.54 kg)</td>
</tr>
<tr>
<td>$k_p$</td>
<td>72,000 lb/ft (1.05 x $10^6$ N/m)</td>
</tr>
<tr>
<td>$I_{w2}$</td>
<td>539.43 slug ft$^2$ (730.91 kg m$^2$)</td>
</tr>
<tr>
<td>$K_{AX}$</td>
<td>$2.26 \times 10^3$ lb ft/rad (3.06 x $10^9$ N·m/rad)</td>
</tr>
<tr>
<td>$W_{APP}$</td>
<td>13,195 lb (58,692 N)</td>
</tr>
</tbody>
</table>
FIGURE 4.13 Effect of Shear Stiffness on the Critical Speed of the 11 DOF Model With Interconnected Wheelsets for Different Values of Bending Stiffness.
of bending stiffness of $1.67 \times 10^6$ ft lb/rad and $3.34 \times 10^6$ ft lb/rad, the curves of critical speed versus shear stiffness exhibit maximum values of 98 and 92 mph, respectively, near a shear stiffness value of $10^5$ lb/ft. The most important result, however, is that for zero bending stiffness, the critical speed increases from about 76 to 198 mph as shear stiffness is increased from about $10^4$ to $10^6$ lb/ft.

Weinstock [47] has analyzed the effects on the kinematic modes of shear and bending spring interconnections between wheelsets of a two axle vehicle. As noted in his study, a change of creep coefficient can affect dramatically the truck behavior. At a value of $k_x = 10^8$ lb/ft, $k_{\theta_L} = 0$, and nominal values for other parameters, the critical speed is about 198 mph for the nominal (full Kalker) values of creep coefficients. When the values of Kalker's creep coefficients are decreased by 50 percent for this configuration, the critical speed drops from 198 to about 70 mph. As actual values of the creep coefficients are expected to vary from about 50 to 100 percent of the full Kalker values in practice, vehicle designers should consider a large range of values of creep coefficients. This large sensitivity to creep coefficient has been found also for freight cars equipped with conventional roller bearing trucks [44].

Two additional cases for $k_{\theta_L} = 0$ were examined. For each of these, the longitudinal primary stiffness was set equal to zero (i.e., $k_z = 0$). Thus, the "total" bending stiffness between axles was zero. In the first of these additional cases $k_x = 0$, and in the second $k_x = 10^8$ lb/ft. In the frist case, two unstable oscillatory modes are present at all speed investigated from 5 to 105 mph with one mode more lightly damped than the other. Each mode
has a frequency very close to the kinematic frequency of a single isolated wheelset. In the second case, where $k_{x_L} = 10^8$ lb/ft, only one unstable oscillatory mode occurs, again at all speed examined from 5 to 105 mph and at a frequency very close to the kinematic frequency of a single isolated wheelset.

To realize the benefits of wheelset interconnection with this configuration, it appears necessary to have a shear interconnection that is effectively rigid as well as some "optimum" value of total inter-axle bending stiffness as determined by the interconnection bending stiffness and the longitudinal primary stiffnesses.

The critical speed was calculated for the vehicle with all three values of bending stiffness at a shear stiffness of $1 \times 10^{10}$ lb/ft. The computed values of critical speed for bending stiffnesses of 0, $1.67 \times 10^6$, and $3.34 \times 10^6$ lb ft/rad are 199, 80 and 61 mph, respectively. Thus, for high levels of shear stiffness, the critical speed for zero bending stiffness shows no signs of decreasing.

The shape of the least-damped mode for the vehicle with interconnected wheelsets changes with increasing shear stiffness. At zero bending stiffness ($k_{x_L} = 0$) with shear stiffness above about $1 \times 10^6$ lb/ft, there are large truck and car body lateral motions with the car body lagging the truck by about 60 degrees. The wheelset lateral motion relative to the truck are each about 20 percent of the magnitude of truck lateral, in phase with one another, and lead truck lateral by about 130 degrees. Truck yaw and car roll are of about the same magnitude and lead truck lateral by about 120 and 60 degrees, respectively. Truck warp is very small and is about 7 percent of truck yaw. Front and rear wheelset yaw relative to the ends
of the sideframes are about one third of the magnitude of truck yaw and are out of phase with each other. Front wheelset yaw lags truck lateral by about 90 degrees. A sketch of this mode shape is shown in Figure 4.14.

At a value of shear stiffness of $1 \times 10^4$ lb/ft and zero bending stiffness ($k_b = 0$), the shape of the least-damped mode is very similar to that of a vehicle with no wheelset interconnection but all other parameters identical.

Both Wickens [23] and Weinstock [47] developed a general, four degree-of-freedom model of two wheelsets interconnected by shear and bending stiffness elements. Weinstock examined the kinematic motions of the system (i.e., inertia and mass terms neglected) and found that the damping ratio of one mode, when plotted against the nondimensional interconnection lateral stiffness for various constant ratios of nondimensional yaw to lateral interconnection stiffness, exhibited a distinct maximum value. He also noted that for small values of either the lateral or yaw interconnection stiffness, the solution was the kinematic oscillation of a single wheelset. Wickens, in an approximate solution to the full set of dynamic equationa, found two oscillatory modes, the frequency of each being the kinematic frequency of a single wheelset. He found that, in general, the damping of both oscillations is positive at low speeds. As speed increases, the damping increases and then decreases until one mode becomes unstable. In the case where the total bending stiffness between axles is zero, Wickens found the critical speed to be zero and the vehicle would undergo a neutrally damped steering oscillation at the kinematic frequency. The value of interconnection shear stiffness did not affect this result. Wickens also examined the limiting case of zero
FIGURE 4.14 Mode Shape of Least-Damped Mode of 11 DOF Model with Interconnected Wheelsets.
bending stiffness and infinite shear stiffness, and found that the solution contains a zero eigenvalue which would imply neutral stability and no preferred equilibrium position.

The results achieved with the 11 DOF model show many similarities with those obtained by Weinstock and Wickens for simpler, generalized models. The cases with zero values for interconnection bending stiffness and longitudinal primary stiffness, in having a slightly unstable oscillation at the wheelset kinematic frequency, are similar to both Weinstock's and Wickens' results. At $k_L = 10^8$ lb/ft, the critical speed of the 11 DOF model increases from zero at a value of total bending stiffness (interconnection bending plus longitudinal primary stiffnesses) of zero, reaches a value of about 198 mph at an intermediate value of total bending stiffness, then decreases to lower values as total bending stiffness increases further. This is very similar to Weinstocks's results for the damping one of the kinematic modes.

Scheffel [24] analyzes both a four degree-of-freedom, two axle vehicle with interconnected wheelsets and a five degree-of-freedom, two axle vehicle identical to the former vehicle but with an added body mass capable of lateral motions. For both a Scheffel's vehicles, the wheelsets were interconnected directly by a shear connection and indirectly through massless sideframes by longitudinal primary stiffness elements. Scheffel found, as did Wickens, that the solution exhibited a zero eigenvalue or neutral stability for infinite interconnection shear stiffness and zero longitudinal primary stiffness. (Wickens' analogous case is that for zero bending stiffness.)
Conclusions

The suspension study reported in this chapter demonstrated that the simpler, three degree of freedom truck model used in earlier parametric studies [44] quite adequately represents the stability characteristics of the conventional roller bearing freight truck. Results from those parametric studies agree quite closely with the results from the 9 degree of freedom truck model used in this study when the primary suspension stiffnesses were on the order of those measured by Brenco [41] for roller bearings.

The versatility of the 11 DOF model was used to investigate several generic truck configurations. Roller bearing truck, plain bearing truck, passenger truck and a modified freight truck configuration with a primary suspension were studied by appropriately varying the primary suspension and warp stiffness parameters. Several conclusions concerning truck design can be drawn from these results.

The addition of primary suspension to a freight truck can with proper choice of suspension parameters, improve stability. When the primary suspension stiffnesses are one third of the nominal (LIMRV) values or greater, the critical speed is higher than for the RBT. However, if the primary suspension values are below one third of the nominal values the critical speed is less than that of the RBT.

The passenger truck provides the greatest stability of the configurations examined in this parametric study. The greatest stability with the passenger truck configuration was achieved when the primary suspension values were reduced to 50% of the nominal (LIMRV) values.
The plain bearing truck (PBT) was found to be slightly less stable than the roller bearing truck. It should be noted, however, that the hunting mode for the PBT model consists of large wheelset motions and small truck and car body motions. Thus, the hunting mode for the PBT model at the critical speed may degrade performance less than the hunting modes for the other configurations in terms of vibration transmission to the car body, but higher rates of wheel and bearing wear may occur. Tests conducted by Seaboard Coast Line Railroad [43] showed that, for a freight car with worn plain bearing trucks, car body and truck motions remained small at speeds that were up to 20 mph more than the critical speed and where sustained hunting oscillations occurred for the same freight car with roller bearing trucks.

The warp stiffness of the truck frame strongly influences freight truck stability, as was shown in earlier parametric studies [44]. At low values of warp stiffness the wheelset motions are relatively unconstrained, resulting in poor stability. At very high values of warp stiffness the truck behaves like a rigid truck, which also has relatively poor stability. An optimum value of warp stiffness lies between these extremes, as far as stability is concerned.

The effect of nominal axle stiffness on stability for all configurations of the 11 DOF model is practically negligible. The critical speed of the nominal model decreases with decreasing axle torsional stiffness and is zero for a stiffness corresponding to an axle diameter of about 2.5 inches. This trend is reported by Doyle and Prause [32] for their model.
The use of interconnected wheelsets can improve freight car stability. From the limited parameter study conducted with the 11 DOF model, it appears that the best choice of interconnection from the point of view of stability (curving was not examined) is that of zero bending stiffness and very high shear stiffness. Although no attempt was made to find an "optimum" combination of interconnection bending stiffness and longitudinal primary stiffness (to yield a "total" inter-axle bending stiffness), an "optimum" value probably does exist as indicated by the trends of the results reported in this chapter. The critical speed for this configuration is quite sensitive to the values of creep coefficients.

It should be noted that the stability of many vehicle configurations was found to be very sensitive to variations on creep coefficients. Thus, any design study should include examination of vehicle performance at a range of creep values.
CHAPTER 5. CAR BODY FLEXIBILITY STUDY

Introduction

A study conducted using a 23 degree-of-freedom "complete vehicle" model is described in this chapter. The effects of lateral and torsional flexibility of the car body model on stability of the model was examined for a hopper car and a flat car.

A primary objective for developing this model was to check the range of validity of the assumptions made for simpler vehicle models that were previously developed in the Freight Car Dynamics research program (i.e., the 5 DOF and 9 DOF models mentioned in previous chapters). From an economic point of view, if the 5 DOF and 9 DOF models were shown to be adequate, they could be used instead of the complete vehicle model. The latter model has 23 degrees-of-freedom and requires more computer time than do either the 5 DOF or 9 DOF models to conduct parametric studies.

The complete vehicle model serves also as a tool for investigating conventional as well as unconventional and existing wheelset, truck and car body designs by adjusting the appropriate parameters. In this regard, the complete vehicle model is more versatile than either the 5 DOF or 9 DOF models.

Description of Model

The complete vehicle model consists of two trucks and a car body. Each of the two trucks is identical to the one considered in Chapter 4. Thus, each truck has nine degrees-of-freedom.
The car body consists of two identical masses connected at a "hinge", where torsional spring and damper combinations act in the horizontal and vertical planes. These suspension elements act at a "hinge height" \( h \), above the centerplates, and may be different from the car body centroid height, \( h_{CG} \). With this configuration, the first lateral and torsional bending modes of the car body can exist.

Five coordinates describe the motion of the car body. They are chosen to be:

1. lateral displacement of the imaginary line connecting the two truck centerplates (this is the lateral displacement of the car body if it were rigid).
2. yaw displacement of imaginary line connecting centerplates (this is the yaw of the car body if it were rigid).
3. angle of deflection of first lateral bending mode.
4. roll displacement of front half car body.
5. roll displacement of rear half car body.

Further assumptions are that the front and rear trucks and the rear wheelsets are identical. Like the model described in Chapter 4, the half-car bodies are considered to roll and move laterally with the bolsters.

As mentioned in Chapter 2, the secondary spring groups of the trucks have finite height and are fixed at the ends. Thus, there is a pure moment exerted across the springs due to the relative motions of the bolster and sideframes.

As discussed before, each truck has nine degrees-of-freedom. These are wheelset lateral, yaw and torsional motions for each of
FIGURE 5.1 Schematic of Complete Vehicle Model.
FIGURE 5.2 Schematic of Flexible Car Body Model (Plan View)
two wheelsets, plus lateral, yaw, and warp of the truck frame. These eighteen degrees-of-freedom, together with the five degree-of-freedom car body described above, give a total of twenty-three degrees-of-freedom for the complete car.

Schematics of the complete vehicle model are shown in Figures 5.1, and 5.2. The dimensions and degrees-of-freedom of the model are given in these figures.

The complete vehicle model was validated against the previously developed 9 DOF model. The 9 DOF model consists of two freight trucks with "ball-join" connections between the wheelsets and side-frames, and a rigid car body. As will be discussed later, results obtained confirm that the complete vehicle model converges to the 9 DOF model by letting the axle torsional stiffnesses, primary suspension stiffnesses, and car body stiffnesses become very large.

**Equations of Motion**

The detailed derivation of the equations of motion for this model are given in Reference [2]. The truck equations are essentially the same as those derived in Appendix B, for the 11 degree of freedom vehicle and are changed only by the definition of the coordinates which describe the motion of the flexible car body. The wheelset equations are identical to those derived in Appendix A.

**Results**

A study was conducted using the complete vehicle model to investigate how the critical speed is affected 1) by variations in car body stiffness for new and curved wheel profiles, 2) by the
amount of lading, and 3) by variations in the height of the car body centroid above the bolster.

The study considered two types of car bodies. They are the L & N 80 ton open hopper car and an ACF 70 ton 89 ft TOFC or flat car. Nominal parameters for each car body for both empty (light) and loaded (heavy) conditions are given in Table 5.1. One-half of the values of creep coefficients predicted by Kalker's linear theory were used. The truck model used was the nominal roller bearing configuration.

A Huemann wheel profile was used with the hopper car configuration for two cases in this study. This profile was designed for wheel/rail contact at a single point for all wheelset lateral positions.

A modified version of the original Heumann profile is shown in Figure 5.3 for comparison with the new and worn profiles shown in Figure 2.3. The kinematic quantities that describe wheel/rail contact with the Heumann profile on new rails were obtained from describing functions developed by Cooperrider et al [15] (see, for example, Figure 2.2). The values of these kinematic quantities are listed in Table 5.1. It should be noted that, in order to obtain a single set of values for the kinematic quantities for the Heumann profile, one must assume an amplitude of wheelset lateral displacement (for this study, an amplitude of 0.3 inches, which is approximately flange clearance for standard gage track, is assumed). A single set of values is not representative of the entire profile. Consequently, the stability of the model is not representative of the effects of the entire profile. To examine more thoroughly the effect on stability of the Heumann profile, one should consider a range of amplitudes of wheelset lateral displacements, and thus, several sets of values
TABLE 5.1

NOMINAL DESIGN PARAMETERS FOR COMPLETE VEHICLE MODEL

(Note: Changes and additions to the nominal design parameters for the 11 DOF model are listed. Other parameters are listed in Table 4.1).

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Wheel/Rail Contact Parameters*

New Wheel Profiles

| \( \alpha_1 \)       | 0.0                                               | 0.0                                              |
|                      | 0.0                                               | 0.0                                              |
| \( \delta_0 \)       | 0.05                                              | 0.05                                             |
|                      | 0.05                                              | 0.05                                             |
| \( a_1 \)            | 0.05                                              | 0.05                                             |
|                      | 0.05                                              | 0.05                                             |
| \( \lambda_1 \)      | 0.05                                              | 0.05                                             |
|                      | 0.05                                              | 0.05                                             |
| \( f_{11} \)         | \( 4.973 \times 10^5 \)                          | \( 7.585 \times 10^5 \)                         |
|                      | \( 2.212 \times 10^6 \)                           | \( 3.375 \times 10^6 \)                         |
| \( f_{12} \)         | \( 2.30 \times 10^3 \)                            | \( 5.051 \times 10^3 \)                         |
|                      | \( 3.118 \times 10^4 \)                           | \( 6.847 \times 10^3 \)                         |

* Creep coefficients are \( \frac{1}{3} \) of values predicted by Kalker's theory.
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Heumann Profiles (Amplitude = 0.3 in) (Not used on flat car model)

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FLAT CAR
FIGURE 5.3 Modified Heumann Wheel Profile.
for the kinematic quantities. A previous study [44] examined the effect on stability of wheels with Heumann profiles on the 9 DOF model. It is the aim of this study to demonstrate the effects on stability of a change in contact geometry. A thorough parametric study of wheels with Heumann profiles is not intended.

The effect of variations in car body stiffness on the stability of the hopper car configuration with new wheels is shown in Figure 5.4. Car body lateral bending and torsional stiffnesses were varied simultaneously so that their ratio remained constant. Over the range of car body stiffness shown in the figure, the critical speed is relatively unaffected by car body stiffness. Speeds obtained with the 9 DOF model for a light and heavy rigid hopper car are indicated in Figure 5.4. It may be seen that, for the nominal car body stiffnesses, the critical speed of the complete vehicle model is about two percent less than the critical speed of 9 DOF model for the light and heavy hopper cars. The critical speeds for one percent of the nominal values of car body stiffness are about ten percent less than those for the rigid car body configurations.

For both heavy and light flexible hopper car body configurations, the mode shapes are similar, and there exist two strongly coupled modes which go unstable. One mode consists of large truck lateral displacements that are about 180 degrees out of phase, and smaller car body lateral displacements. Car body roll and yaw displacements are about equal in magnitude, and half-car body roll displacements are about 180 degrees out of phase. This mode is the first to go unstable. At a slightly higher speed (about 5 mph higher), a second mode goes unstable. This mode consists of large truck lateral displacements and smaller car body lateral displacements. Car body
FIGURE 5.4 Effect of Car Body Stiffness on Critical Speed of the Complete Vehicle Model (Hopper Car with New Wheels)

PERCENTAGE OF NOMINAL CAR BODY STIFFNESS
(o-Light Car; △-Heavy Car)
yaw displacements are small. The displacements of the two
trucks are in phase and the half-car bodies move "together" like a rigid

car body. As the car body stiffness is increased, the amplitude of out-
of-phase car body roll displacements decreases while the amplitude of in-
phase car body roll displacements increases. Lateral bending displac­
ements are small. For both modes, the trucks have the same mode shape as
for the roller bearing truck (RBT) configuration described in Chapter 4.
The least damped modes for the complete vehicle (flexible car) model are
illustrated in Figure 5.5.

The least-damped modes for the 9 DOF (rigid car) model differ
somewhat from those for the complete vehicle model. These modes are
shown in Figure 5.6. A major distinction between the two models is
that the 9 DOF includes a rigid car body, while the complete vehicle
model has a flexible car body. This distinction probably accounts
for the differences in mode shapes.

Unlike the flexible car configurations, the light and heavy
rigid car configurations have different least-damped modes. The
least-damped mode for the rigid heavy car is similar to that for
the light and heavy flexible cars, but has small car body roll that
corresponds to in-phase roll displacements of the flexible car body.
The least-damped mode for the rigid light car consists of large
in-phase displacements of the trucks (similar to that shown in
Figure 5.5 (b)), and large car body lateral displacements that are
in phase with the lateral displacements of the trucks.

The effect of car body lateral bending and torsional stiffness
(varied together) on critical speed is shown in Figure 5.7 for the
a) Least-Damped Mode

(Light and Heavy Cars)

b) Second Mode To Go Unstable

(Light and Heavy Cars)

FIGURE 5.5 Least-Damped Modes for Flexible Hopper Car Configurations (New and Heumann Wheel Profiles).
FIGURE 5.6 Least-Damped Modes for Rigid Hopper Car Configurations with New Wheels.
FIGURE 5.7 Effect of Car Body Stiffness on Critical Speed of the Complete Vehicle Model (Hopper Car with Heumann Wheel Profiles)
light and heavy flexible and rigid cars with Heumann wheel profiles for an assumed amplitude of wheelset lateral displacement of 0.3 inches. For the nominal car body stiffness, the critical speeds of the light and heavy flexible hopper cars with Heumann wheel profiles are about ten and six percent higher, respectively, than the critical speeds for the same configurations with new wheels. The critical speeds of the light and heavy rigid cars with Heumann wheel profiles are about three percent higher than those for the rigid car with new wheels. The critical speeds for the nominally flexible light and heavy cars with Heumann wheel profiles are about eight and four percent higher respectively, than the light and heavy rigid cars. Cooperrider and Law [44] show that the critical speed may be increased significantly by using Heumann wheels profiles on both light and heavy vehicles. They also show that Heumann wheel profiles have a stronger effect on the critical speed of heavy cars than for light cars due to the increased gravitational stiffness effects that result with Heumann profiles.

Like the new wheel cases, car body stiffness has a small effect on the critical speeds of the hopper car configurations with Heumann wheel profiles for the range of stiffnesses considered. A decrease in car body stiffness has a slightly stronger effect on the critical speed of the heavy car with Heumann wheel profiles than with new wheels.

The shape of the least-damped mode for the light and heavy flexible cars with Heumann wheel profiles and with new wheels are similar. The least damped mode for the light rigid car with Heumann wheel profiles is identical to that with new wheels.
The least-damped mode of the heavy rigid car with Heumann wheel profiles differs from that with new wheels. With Heumann wheel profiles, the least-damped mode is similar to the "second mode to go unstable" (Figure 5.5 (b)) for the light and heavy flexible cars with new wheels. These modes are illustrated in Figure 5.8.

It is interesting that car body weight and the change in wheel/rail geometry that was prescribed have a small effect on the least-damped mode for the flexible car configurations. For the rigid car configurations, however, the mode shapes for the light and heavy cars are distinct. The least-damped mode for the light rigid car has typically larger car body lateral motions than that for the heavy rigid car, and is affected to a small degree by the change in wheel/rail geometry. The least-damped modes for the heavy rigid car with new wheels and with Heumann wheel profiles differ significantly. For new wheels, the trucks are about 180 degrees out of phase. For Heumann wheel profiles, the trucks are nearly in phase.

The stabilizing effect of the Heumann wheels can be attributed to the "contact angle difference" effect provided by the contact geometry. As discussed in Chapter 2, the gravitational stiffness force is a function of the difference in contact angles between the left and right wheels, as well as the axle load and wheelset roll angle. For new wheels, the difference in contact angles is zero. For Heumann wheel profiles, the difference in contact angles is a positive non-zero function of wheelset lateral displacement. Consequently, the stabilizing effect of the gravitational stiffness force is stronger for Heumann wheel profiles than for new wheels.
FIGURE 5.8 Least-Damped Modes for Rigid Hopper Car Configurations with Heumann Wheel Profiles.
In the tread region, the effective conicity is larger for Heumann wheels than for new wheels. As mentioned previously, wheel conicity is a destabilizing effect. For the hopper car configurations with Heumann wheel profiles, the gravitational stiffness effect probably overcomes the conicity effect.

As shown in Figures 5.4 and 5.7, the critical speeds for the heavy configurations are as much as 75 percent higher than the critical speed for the light car configurations. This stabilizing trend was observed during tests conducted by Seaboard Coast Line Railroad [42]. This trend can be attributed to an increase in the gravitational stiffness force resulting from an increase in the axle load. The creep coefficients are also larger for the heavy car. As shown in our previous study [44] and by Doyle and Prause [32], among others, increases in creep coefficients may be stabilizing. As our results indicate [44], however, critical speed is not a monotonic function of the creep coefficients. Thus, it would be erroneous to generalize that increased creep coefficients are always a stabilizing effect.

The effect of car body torsional stiffness on critical speed with constant "stiff" lateral bending stiffness is shown for the flat car (TOFC) configuration with new wheels in Figure 5.9. For the light flat car, the critical speed decreases by about 1.6 percent as the value of car body torsional stiffness is increased from one percent of the nominal value to ten times the nominal value.
FIGURE 5.9 Effect of Car Body Torsional Stiffness on Critical Speed of the Complete Vehicle Model (Flat Car with New Wheels).
For the heavy flat car, however, the critical speed increases from about 87 mph to 93 mph as the car body torsional stiffness is increased from one percent of nominal to nominal stiffness. The critical speed decreases to about 73 mph as car body torsional stiffness is increased from nominal to ten times the nominal torsional stiffness. Thus, there may exist some "optimum" value of car body torsional stiffness at which the critical speed is a maximum.

The critical speeds for the light and heavy flat cars are greater than the critical speeds for the light and heavy hopper cars, respectively. This is probably due to the greater distance between truck centers (66.08 ft for the flat car versus 33.7 ft for the hopper car). The effect on stability of truck center distance may be analogous to the same effect of truck wheelbase as shown by Clark and Law [39], among others.

Like the hopper car configuration, the critical speed of the heavy flat car is higher than the light car. As mentioned previously, this may be attributed to the stronger stabilizing effect of the gravitational stiffness for larger axle loads.

For the unstable mode of the light car, car body roll displacements are the largest angular displacements, and car body lateral displacements are the largest linear displacements of the vehicle. Car body roll displacements are in phase and lead car body lateral displacements by about 180 degrees. The two trucks move in phase. A second mode goes unstable at a speed just above the critical speed. This mode consists of out-of-phase roll displacements of the front and rear half-car bodies and out-of-
phase lateral displacements of the front and rear trucks. The relative magnitudes of displacements are similar to those for the first mode to go unstable.

For the unstable mode of the heavy flat car, truck lateral displacements are the largest linear displacements, and truck yaw displacements are the largest angular displacements of the vehicle. Truck lateral displacements are about 180 degrees out of phase, and car body lateral displacements lag from truck displacements by about 35 degrees. Roll displacements of the two half-car bodies have approximately the same magnitude as truck yaw displacements, and increase in magnitude with decreasing car body torsional stiffness. The half-car bodies are about 180 degrees out of phase with each other, and front half-car body roll displacements lag front truck lateral displacements by about 30 degrees.

To examine the effect on critical speed of car body centroid height about the "hinge", four configurations were compared. These are 1) a nominal TOFC car that is uniformly loaded; 2) a nominal TOFC car loaded only in the top half of the car body, i.e., "top-loaded"; 3) a nominal COFC car that is uniformly loaded; and 4) a nominal COFC car loaded only in the bottom half of the car body, e.e., "bottom-loaded". (The TOFC and COFC configurations differ only by the roll moments of inertia and by the centroid heights). These four configurations represent a range of centroid heights above the "hinge" of from 2.43 to 9.27 feet. In all cases, the loads were the same; load "density" for cases 2 and 4 was assumed to be twice that for cases 1 and 3.
The critical speeds of these four configurations are shown in Figure 5.10, where critical speed is plotted versus centroid height above the "hinge". The roll moments of inertia used for each configuration are also indicated in the figure. As shown in the figure, the critical speed is slightly lower for flat car configurations having a higher centroid location above the "hinge".

The critical speed of the uniformly-loaded TOFC car is about two percent higher than the car with load distributed throughout only the top half of the car body. The critical speed of the uniformly-loaded COFC car is about 3.5 percent lower than the "bottom-loaded" COFC car. Thus, significant changes in the location of the loads have a small effect on the critical speed of the flat car configurations.

The mode shapes for the four configurations are similar to those for the nominal heavy TOFC car that was discussed previously, but with different magnitudes of car body roll displacements. The magnitudes of out-of-phase car body roll displacements for the "top-loaded" TOFC car are about ten percent higher than those for the uniformly loaded TOFC car, and roll displacements for the "bottom-loaded" COFC car are about ten percent lower than those for the uniformly-loaded COFC car.

Tse [30] found with his model that the car body centroid height influenced significantly the speed at which sustained car body rocking on the bolsters occurred due to excitation from half-staggered rail joints. His model did not consider wheelset yaw degrees-of-freedom. Consequently, hunting motions could not occur.
FIGURE 5.10 Effect of Car Body Centroid Height on Critical Speed of Complete Vehicle Model (Heavy Flat Car with New Wheels)

○ TOFC - Loaded Uniformly ($I_{c3} = 3.12 \times 10^5$ slug-ft$^2$)
● TOFC - Loaded in Top-Half Only ($I_{c3} = 3.68 \times 10^5$ slug-ft$^2$)
▲ COFC - Loaded Uniformly ($I_{c3} = 2.39 \times 10^5$ slug-ft$^2$)
▲ COFC - Loaded in Bottom-Half Only ($I_{c3} = 1.80 \times 10^5$ slug-ft$^2$)
Car body rocking relative to the bolsters probably should be included in future models to assess the influence on critical speed. As mentioned in Chapter 2, it is certainly important to include a car body rocking model to examine forced response to excitation from the track. In stability analyses, the model used in these studies is probably adequate for most freight car configurations. For "extreme" cases, e.g., a light car with high center of gravity, relative roll motions between the car body and bolsters may be significant for small disturbances from equilibrium, and may be a factor in determining stability.

Conclusions

The primary objective of the study reported in this chapter was to determine whether car body flexibility strongly affects hunting stability. The critical speed, frequency and damping results obtained for the 80 ton hopper car and unloaded 70 ton flat car are very similar to those obtained for the simpler, 9 DOF model. However, the mode shapes obtained when car flexibility is included differ considerably from the rigid car modes. Also, for the heavy flat car, there was an optimum value of torsional bending stiffness that maximizes critical speed. Thus it appears that car body flexibility may be neglected for stability studies of relatively stiff vehicles, but should be included for more flexible car bodies.

The brief parametric studies conducted with the complete vehicle model revealed several points of interest to vehicle
designers. The height of the car body centroid has a small effect on the critical speed of the flat car configuration. It has been shown by others that the occurrence of sustained car body rocking relative to the bolsters due to excitation from the track can be influenced strongly by car body centroid height. A car body rocking model may be valuable in more thoroughly determining the effect on stability of car body centroid height.

The critical speeds for the nominal light and heavy flat car configurations are higher than those for the nominal light and heavy hopper car configurations. This is probably due to the greater distance between truck centers for the flat car configurations.

Although the critical speeds for the hopper car configuration with Heumann wheel profiles were as much as 10% higher than those for a vehicle with new wheel profiles these results should be considered with caution. The critical speeds are highly sensitive to variations in creep coefficient and suspension damping values, and these effects do not act in the same way for vehicles with different wheel profiles. For example, the parametric study conducted earlier [44], showed that when the same hopper car stability was computed using the full values of the creep coefficients predicted by Kalker's theory, the new wheel vehicle critical speed was about twice that given here at 50% of Kalker's values, while the Heumann wheel vehicle critical speed remained at the value computed here. In addition, the Heumann wheel result shown here was computed for one value of wheelset lateral amplitude (0.3 inches which corresponds to a contact point on the wheel in the throat of the flange). Critical speeds at other values of lateral
amplitude are lower. Thus, an accurate picture of vehicle stability with differing wheel profiles can only be obtained from a reasonably extensive parameter study.
CHAPTER 6. SUMMARY AND CONCLUSIONS

Detailed conclusions of the studies of the torsionally flexible wheelset, the eleven degree of freedom half car model, and the twenty three degree of freedom flexible car model may be found at the ends of Chapters 3, 4, and 5 respectively. An overall summary of the work together with a distillation of the detailed conclusions of Chapters 3, 4, and 5 are presented in this chapter.

Development of Models

The general twenty-three degree of freedom model described in this report was developed and used to check the limits and range of validity of simpler models having fewer degrees of freedom. Subsequently, the model was employed to examine the effects on stability of a variation in vehicle design parameters.

The general vehicle model was developed in stages. In the first stage, attention was directed to modeling a torsionally flexible wheelset. Secondly, a generalized truck model incorporating two of these wheelsets together with a pseudo car body was assembled. Finally, the complete vehicle model was obtained by incorporating two of the generalized truck models together with a flexible car body.

The versatility of the complete vehicle model is provided by the torsionally flexible wheelsets, the primary suspension groups, the "warping" truck frame, and the flexibility of the car body. With these features, a wide variety of conventional and unconventional freight car designs may be investigated.

Validation of Models

The validity of models described in the proceeding chapters was checked by prescribing limiting cases that could be represented by other existing
models with fewer degrees of freedom. For example, for very large values of axle torsional stiffness, the three degree of freedom wheelset exhibits stability characteristics that are virtually identical to those of a two degree of freedom wheelset with a rigid axle.

The twenty three degree of freedom model was also checked against a nine degree of freedom model of a freight car with a rigid car body and roller bearing trucks by increasing the stiffnesses of the car body, primary suspension elements, and axles to very large values. Each of the limiting cases checked well with existing, simpler models in terms of mode shapes, frequencies, and damping ratios.

The range of validity of the assumptions of torsionally rigid wheelsets, "ball joint" wheelset/sideframe connections, and a rigid car body were determined by the studies described in Chapters 3, 4, and 5. The assumption of torsionally rigid axles is reasonable for stability studies as the effect on critical speed of axle flexibility for conventional wheelsets is practically nil. For conducting stability analyses of roller bearing freight trucks, the assumption of "ball joint" wheelset/sideframe connections is reasonable. With values of the primary suspension stiffnesses on the order of those measured for actual roller bearing trucks, the stability characteristics of the more general truck model were very similar to those of the simpler truck model with ball-joint connections. Finally, the critical speed of the vehicle model with a nominally flexible car body (using the two-mass car body approximation) is within two percent of that of the vehicle with a rigid car body. However, the shapes of the least damped modes were different for these models. Thus, it is probably preferable to use vehicle models that include car body flexibility in the final stages of the design process.
Application of Models

Three Degree of Freedom Wheelset Model

The studies of the effects of design parameter variations were conducted in three phases. The initial study focused on the effects of axle torsional stiffness on the lateral stability of the three degree of freedom wheelset model. As discussed in Chapter 3, the critical speed of the wheelset model is increased significantly by decreasing the axle torsional stiffness to a value corresponding to an axle diameter of about three inches. However, studies using the more realistic eleven degree of freedom vehicle model indicate that critical speed always decreases with decreasing axle torsional stiffness, reaching a value of zero for axle stiffness below that corresponding to a 2.5 inch diameter axle.

Studies of the wheelset model equipped with either independently rotating or cylindrical wheels show that the guidance mechanism is lost for these configurations. Thus it is doubtful whether acceptable performance would be achieved on either tangent or curved truck.

Eleven Degree of Freedom Half Car Model

The second phase of the design parameter study was conducted with the eleven degree of freedom model. It was shown that various truck configurations could be obtained by adjusting the stiffnesses and damping of the truck frame in warp and the primary suspension elements. Examples of these configurations are: roller bearing truck, plain bearing truck, rigid truck, and rigid truck frame with primary suspension.

It was shown that adding a primary suspension to the roller bearing truck configuration could increase the critical speed. The critical speed can be increased further by stiffening the truck frame. However, increasing
or decreasing the truck frame warp stiffness from the nominal value for trucks without a primary suspension decreased the critical speed.

A simple, generic model of a roller bearing truck with interconnected wheelsets was developed and studied. Based on the results of this brief study, it was found that large increases in critical speed could be obtained for large values of interconnection shear stiffness. However, the "total" bending stiffness, i.e. the combined interconnection bending stiffness and longitudinal primary stiffness must be carefully chosen. Results also indicated that the stability of the configuration with interconnected wheelsets was quite sensitive to the values used for the creep coefficients.

As also shown in the wheelset study, results obtained with the eleven degree of freedom model indicate that guidance or self-centering capability of the wheels is lost when cylindrical or independently rotating wheels are used. Use of cylindrical or independently rotating wheels would therefore result in increased wheel flange and rail wear and poor performance on both tangent and curved track.

Twenty Three Degree of Freedom Car Model

In the third phase of the parameter study, the complete vehicle model was used to examine the effects on stability of variations in the torsional and lateral bending flexibility of both a hooper and a flat car. The influence of center of gravity height on stability was examined for a loaded flat car.

Variations in car body flexibility had very little effect on the stability of the hooper car and the light flat car. There was a slight decrease in the critical speed of the heavy flat car with changes of the torsional stiffness from the nominal value.
Critical speed decreased slightly as the height of the center of gravity of the loaded flat car was increased from 2.43 to 9.27 feet above the centerplate.

One of the most important results of this phase is that the critical speeds predicted for vehicles with either rigid or flexible car bodies did not differ appreciably. However, shapes of the least damped mode for vehicles with rigid car bodies differed from those obtained for the vehicles with the flexible car bodies. Thus, to obtain an accurate representation of the mode shapes, a flexible car body model should be used.

On-Going Studies

Future reports will describe further stability studies conducted with the nominal hopper car with roller bearing trucks. These studies are in support of the validation effort associated with the field tests recently completed by the Association of American Railroads on the Union Pacific Railroad.

Additional reports will cover the development of the models and associated analyses for lateral forced response and curving performance of freight cars. A report also will be issued defining and describing the simpler models developed to study the stability behavior of rail freight vehicles. For many configurations (including the roller bearing truck), these simpler models adequately describe the stability behavior and are significantly more economical to use than the 23 DOF model presented in this report.

Conclusions

Models of rail vehicle have been developed to investigate lateral stability. The most general of these is a twenty three degree of freedom
model that includes approximations of those effects of car body flexibility likely to influence vehicle stability. This vehicle model also employs a very general truck model that may be used in modeling trucks with flexible frames, interconnected wheelsets, and primary suspension. This truck model may also be specialized by appropriate choice of parameters to configurations representative of passenger trucks and plain and roller bearing freight trucks.

As previously discussed, the main purpose of this report and the work described herein is to present the development of models suitable for lateral stability analyses of rail vehicles rather than to conduct extensive parameter studies. However, brief parameter studies were conducted mainly for the purpose of investigating modeling assumptions. As the results of this work have shown, stability is affected by many design parameters. Variations in a particular parameter, such as truck warp stiffness for example, may increase or decrease critical speed depending on the values of all the other parameters. Thus it is dangerous to try to say for rail vehicles in general, that "increases in parameter A will increase (or decrease) critical speed". It is possible to identify parameters that can affect strongly stability. The exact manner in which they do affect stability, however, is highly configuration-dependent.

Parameters that appear to affect strongly the stability of freight cars with roller bearing trucks are: wheel profile shapes, creep coefficient values, truck warp stiffness, suspension damping, and load conditions. It should be noted that these are most of the parameters that can be expected to vary for a vehicle with roller bearing trucks similar to existing configurations.
In Appendix A, the model of the flexible wheelset is developed. Included in the development are the derivation of the linearized creep forces and moments and the development and discussion of the gravitational stiffness forces.

The equations of motion for both the 11 degree of freedom model and the 23 degree of freedom model discussed in Chapters 4 and 5, respectively, are developed in detail in [2]. Consequently, these developments are not repeated here. Rather, the resulting matrix equations for these models are presented in Appendices B and C. The equations of motion for each model may be expressed in the form

\[(s^2 \[M] + s \[C] + \[K]) \{X(s)\} = \{0\},\]

where \(\{X(s)\}\) is the displacement vector of order \(n\) (where \(n\) is the number of degrees of freedom of the model), and \([M]\), \([C]\), and \([K]\) are the \(n \times n\) mass, damping, and stiffness matrices, respectively, and \(s\) is a complex eigenvalue.

As mentioned previously, a numerical solution of the eigenvalue/eigenvector problem was used in the parametric studies described in the preceding chapters. A detailed description of the solution method is given by Wilkinson [49] and by Marcotte [50].
A. DERIVATION OF EQUATIONS OF MOTION FOR FLEXIBLE WHEELSET MODEL

Kinematics

Dimensions

Schematics of the wheelset model developed here are shown in Figures 3.1 and 3.2. The dimensions of the model and the degrees of freedom which describe its motion are indicated in these figures.

Axes

The axis systems used for this model are shown in Figure A.1. The "triple-primed" or space-fixed axis system is fixed to the track centerline. The $\hat{i}'''$, $\hat{j}'''$ and $\hat{k}'''$ unit vectors are directed perpendicular to the track centerline in the plane of the track, normal to the plane of the track, and coincident with the track centerline, respectively. The $\hat{i}'''$ vector points to the left, $\hat{j}'''$ points up, and $\hat{k}'''$ points forward along the track.

The "primed" or wheelset axis system is fixed to the wheelset centroid. The $\hat{i}'$ axis is coincident with the axle centerline, $\hat{k}'$ is parallel to the rail plane and is directed forward in the direction of motion, and $\hat{j}'$ is normal to the $\hat{i}'$ and $\hat{k}'$ vectors and points up.

The wheelset and space-fixed axes are related by the expressions

$$
\begin{align*}
\begin{bmatrix}
\hat{i}''' \\
\hat{j}''' \\
\hat{k}'''
\end{bmatrix} &= 
\begin{bmatrix}
\cos \theta \cos \phi & -\cos \theta \sin \phi & \sin \theta \\
\sin \phi & \cos \phi & 0 \\
-\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\hat{i}' \\
\hat{j}' \\
\hat{k}'
\end{bmatrix}
\end{align*}
$$
FIGURE A.1 Axis Systems for Wheelset Model.
Displacement, Velocity, and Acceleration

The following quantities are defined in terms of the described axis systems.

The position, velocity and acceleration of the wheelset center of gravity are:

\[
\mathbf{r}_\text{CG} = x \mathbf{i} + y \mathbf{j} + (z + z_o) \mathbf{k},
\]

(A.2)

\[
\ddot{\mathbf{r}}_\text{CG} = \ddot{x} \mathbf{i} + \ddot{y} \mathbf{j} + (\ddot{z} + \ddot{\nu}) \mathbf{k},
\]

(A.3)

and

\[
\dddot{\mathbf{r}}_\text{CG} = \dddot{x} \mathbf{i} + \dddot{y} \mathbf{j} + \dddot{z} \mathbf{k},
\]

(A.4)

respectively.

The angular velocity of the left wheel is

\[
\overline{\mathbf{\omega}}_L = (\Omega + \dot{\omega}_L) \mathbf{i} + e \mathbf{j} + \dot{\phi} \mathbf{k}
\]

or, using Equations (A.1) and making small angle assumptions,

\[
\overline{\mathbf{\omega}}_L = (\Omega + \dot{\omega}_L) \mathbf{i} + (\phi \Omega + \dot{\phi}) \mathbf{j} + (\dot{\phi} - \phi \Omega) \mathbf{k}
\]

(A.5)

Similarly, the angular velocity of the right wheel is

\[
\overline{\mathbf{\omega}}_R = (\Omega + \dot{\omega}_R) \mathbf{i} + (\phi \Omega + \dot{\phi}) \mathbf{j} + (\dot{\phi} - \phi \Omega) \mathbf{k}
\]

(A.6)
The angular velocity of the wheelset is given by
\[
\overline{\omega} = \frac{1}{2}(\overline{\omega}_R + \overline{\omega}_L) = \Omega \hat{i}'' + (\phi \Omega + \dot{\phi})\hat{j}'' + (\dot{\phi} - \theta \Omega)\hat{k}'',
\]
or, in terms of the wheelset body axes and for small displacements,
\[
\overline{\omega} = \Omega \hat{i} + \dot{\phi} \hat{j} + \dot{\phi} \hat{k}.
\] (A.7)

Angular Momenta

The wheelset is assumed to be symmetric, and the moment of inertia dyadic for the wheelset about the center of gravity is
\[
\overline{I}_w = I_{w1} \hat{i}'' + I_{w2} \hat{j}'' + I_{w2} \hat{k}''.
\] (A.8)

Since the axle is assumed massless, the weight of the wheelset is divided equally between the two wheels. Thus, the moment of inertia dyadic for each wheel about its centroid is
\[
\frac{1}{2} \overline{I}_{w1} \hat{i}'' + \frac{1}{2} I_{w2} (\hat{j}'' + \hat{k}'').
\] (A.9)

The angular momentum of the entire wheelset about the wheelset centroid is then
\[
\overline{H}_G = \overline{I}_w \cdot \overline{\omega} = I_{w1} \Omega \hat{i} + I_{w2} \dot{\phi} \hat{j} + I_{w2} \dot{\phi} \hat{k}.
\]
or, in terms of the space-fixed axes,
\[
\overline{H}_G = I_{w1} \Omega \hat{i}'' + (I_{w1} \Omega \phi + I_{w2} \dot{\phi})\hat{j}'' + (I_{w2} \dot{\phi} - I_{w1} \Omega \phi)\hat{k}''.
\] (A.10)

The angular momentum of the left wheel about its centroid is
\[
\overline{H}_{GL} = \frac{1}{2} \overline{I}_w \overline{\omega}_L + m_w [\frac{d}{dt}(a\hat{i}')]
\] (A.11)
The angular momentum of the right wheel about its centroid is
\[ \overline{H}_R = \frac{1}{2} I_{w1} \overline{\omega}_R + \left( -a \hat{i} \right) \times m_w \left[ \frac{d}{dt} \left( -a \hat{i} \right) \right] \]
\[ = \frac{1}{2} I_{w1} \left( \omega + \dot{\theta} \right) \hat{i} + \left( \frac{1}{2} I_{w2} \dot{\phi} - m_w a^2 \delta \right) \hat{j} \]
\[ + \left( \frac{1}{2} I_{w2} \phi - m_w a^2 \dot{\phi} \right) \hat{k}. \]  
(A.12)

The time rates of change of angular momentum are given by the relation
\[ \overline{H} = \frac{\partial}{\partial t} (\overline{H}) + \overline{\omega}_{\text{axes}} \times \overline{H}. \]
where \( \overline{\omega}_{\text{axes}} \) is the angular velocity of the axis system. The partial derivative notion refers to the fact that the unit vectors are assumed constant.

Then, for the wheelset about its centroid,
\[ \overline{H}_G = \frac{\partial}{\partial t} (\overline{H}_G) + \left( \overline{\omega} - \overline{\Omega} \right) \times \overline{H}_G \]
\[ = \left( I_{w2} \ddot{\phi} + I_{w1} \omega \dot{\phi} \right) \hat{j} + \left( I_{w2} \ddot{\phi} - I_{w1} \omega \dot{\phi} \right) \hat{k}. \]  
(A.13)

For the left wheel about its centroid,
\[ \overline{H}_L = \frac{\partial}{\partial t} (\overline{H}_L) + \overline{\omega}_L \times \overline{H}_L \]
\[ = \frac{1}{2} I_{w1} \ddot{\omega}_L \hat{i} + \left( \frac{1}{2} I_{w2} - m_w a^2 \right) \dot{\omega} - \omega \delta \left[ \frac{1}{2} \left( I_{w2} - I_{w1} \right) - m_w a^2 \right] \hat{j} \]
\[ + \left( \frac{1}{2} I_{w2} - m_w a^2 \right) \dot{\phi} + \omega \delta \left[ \frac{1}{2} \left( I_{w2} - I_{w1} \right) - m_w a^2 \right] \hat{k}. \]  
(A.14)

For the right wheel about its centroid,
\[ \overline{H}_R = \frac{\partial}{\partial t} (\overline{H}_R) + \overline{\omega}_R \times \overline{H}_R \]
\[ = \frac{1}{2} I_{w1} \ddot{\omega}_R \hat{i} + \left( \frac{1}{2} I_{w2} - m_w a^2 \right) \dot{\omega} - \omega \delta \left[ \frac{1}{2} \left( I_{w2} - I_{w1} \right) - m_w a^2 \right] \hat{j} \]
\[ + \left( \frac{1}{2} I_{w2} - m_w a^2 \right) \dot{\phi} + \omega \delta \left[ \frac{1}{2} \left( I_{w2} - I_{w1} \right) - m_w a^2 \right] \hat{k}. \]  
(A.15)
Forces and Moments

The free-body diagrams for the wheelset and each wheel are shown in Figures A.2 and A.3. The forces and moments acting on the wheelset are now defined.

Creep Forces and Moments

As mentioned in Chapter II, creep forces are the tangential forces generated between the wheels and rails due to the difference in the rates of strain, or creepage, in the contact region. Thus, the total creep force acting in the contact region can be resolved into lateral and longitudinal force components. There is also a "spin" moment acting about the normal to the contact plane.

By defining axis systems at each wheel/rail interface, it follows that

\[ \vec{F}_{L} = F_{\text{lat}_L} \hat{e}_{1L} + F_{\text{long}_L} \hat{e}_{3L} \]
\[ \vec{M} = M_{L} \hat{e}_{2L} \]
\[ \vec{F}_{R} = F_{\text{lat}_R} \hat{e}_{1R} + F_{\text{long}_R} \hat{e}_{3R} \]

where \( F_{\text{lat}_L} \), \( F_{\text{long}_L} \), and \( F_{\text{lat}_R} \), \( F_{\text{long}_R} \) are the lateral and longitudinal components of the total creep force at the left and right wheels. The terms \( M_{L} \) and \( M_{R} \) are the spin moments for left and right wheels. For the left wheel, the \( \hat{e}_{1L} \), \( \hat{e}_{2L} \), \( \hat{e}_{3L} \) unit vectors are defined with respect to the plane of contact as tangent and in the lateral direction, normal, and tangent and in the longitudinal direction, respectively. The \( \hat{e}_{1R} \), \( \hat{e}_{2R} \), \( \hat{e}_{3R} \) unit vectors are defined similarly for the right
FIGURE A.2 Forces and Moments Acting on Wheelset Model.
FIGURE A.3. Forces and Moments Acting on Wheelset Model (Right Side View).
wheel contact point. Figure A.4 shows the configurations of these axes, which are related the fixed axes by the expressions

\[
\begin{bmatrix}
\hat{e}_{1R} \\
\hat{e}_{2R} \\
\hat{e}_{3R}
\end{bmatrix} = \begin{bmatrix}
+1 & -(\delta_{R} - \phi) & -\theta \\
(\delta_{R} - \phi) & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{e}_{1} \\
\hat{e}_{2} \\
\hat{e}_{3}
\end{bmatrix} \tag{A.16}
\]

\[
\begin{bmatrix}
\hat{e}_{1L} \\
\hat{e}_{2L} \\
\hat{e}_{3L}
\end{bmatrix} = \begin{bmatrix}
1 & (\delta_{L} + \phi) & -\theta \\
(-\delta_{L} - \phi) & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{e}_{1} \\
\hat{e}_{2} \\
\hat{e}_{3}
\end{bmatrix} \tag{A.17}
\]

From Kalker's linear theory [11], the creep forces and moments are, where \(\hat{r}_{L} \) and \(\hat{r}_{R} \) are the difference in velocities of the wheels and rails at the contact points on the left and right sides,

\[
F_{L} = \left[ \frac{f_{11L}}{V} (-\hat{r}_{L} \cdot \hat{e}_{1L}) + \frac{f_{12L}}{V} (-\hat{\omega}_{L} \cdot \hat{e}_{2L}) \right] \hat{e}_{1L}
\]

\[
+ \left[ \frac{f_{33L}}{V} (-\hat{r}_{L} \cdot \hat{e}_{3L}) \right] \hat{e}_{3L} \tag{A.18}
\]

\[
M_{L} = \left[ \frac{f_{12L}}{V} (-\hat{r}_{L} \cdot \hat{e}_{1L}) + \frac{f_{22L}}{V} (-\hat{\omega}_{L} \cdot \hat{e}_{2L}) \right] \hat{e}_{2L} \tag{A.19}
\]

for the left wheel, and

\[
F_{R} = \left[ \frac{f_{11R}}{V} (-\hat{r}_{R} \cdot \hat{e}_{1R}) + \frac{f_{12R}}{V} (-\hat{\omega}_{R} \cdot \hat{e}_{2R}) \right] \hat{e}_{1R}
\]

\[
+ \left[ \frac{f_{33R}}{V} (-\hat{r}_{R} \cdot \hat{e}_{3R}) \right] \hat{e}_{3R} \tag{A.20}
\]

\[
M_{R} = \left[ \frac{f_{12R}}{V} (-\hat{r}_{R} \cdot \hat{e}_{1R}) + \frac{f_{22R}}{V} (-\hat{\omega}_{R} \cdot \hat{e}_{2R}) \right] \hat{e}_{2R} \tag{A.21}
\]

for the left wheel, where the subscripted "f"'s are the creep coefficients.
FIGURE A.4 Axis Systems at Wheel/Rail Contact Points
As mentioned previously, the creep coefficients are nonlinear functions of the normal load and the wheel/rail contact geometry. Thus, they will vary in magnitude due to dynamic loading and changes in the wheel/rail geometry. When the equations are linearized, these coefficients appear in the equations as the values that exist when the wheelset is in the equilibrium condition. Since the wheelset is symmetrical, the values for left and right wheels are equal and 
\[
f_{11L} = f_{11R} = f_{11}, \quad f_{12L} = f_{12R} = f_{12}, \quad f_{22L} = f_{22R} = f_{22}, \quad \text{and} \quad f_{33L} = f_{33R} = f_{33}.
\]

The term \( \vec{v}_{LC} \) is defined as the difference in velocities between a point on the left wheel in contact with the left rail, and a point on the left rail in contact with a point on the left wheel. The term \( \vec{\omega}_{LC} \) is the net angular velocity at the point of wheel/rail contact on the left side. The total creepage on the left wheel is \( \vec{r}_{Lc} / V \), and the total spin creepage on the left wheel is \( \vec{\omega}_{LC} / V \).

For this model, the rails are considered perfectly rigid. Thus, the creepage is defined as the velocity of the contact point on the wheel in contact with the rail. The absolute velocity of the rail is identically zero. Consequently, \( \vec{u}_{LC} = \vec{u}_L \) and \( \vec{\omega}_{RC} = \vec{\omega}_R \).

The position of the left wheel point of contact is
\[
\vec{r}_{LC} = \vec{r}_{CG} + (a + \Delta_{l\omega}) \hat{i}' - r_L \hat{j}',
\]
where \( \Delta_{l\omega} \) is the lateral displacement from equilibrium of the contact point on the left wheel.
Then, the velocity of this contact point is
\[
\dot{r}_{Lc} = \dot{r}_{CG} + \omega_L \times [(a + \Delta_{\omega})\hat{i}' - r_L \hat{J}']
\]
\[
= \dot{x} \hat{i}''' + \dot{y} \hat{j}''' + (\dot{z} + V) \hat{k}'''
\]
\[
+ \begin{pmatrix}
\hat{i}' & \hat{j}' & \hat{k}' \\
\hat{e} & \hat{d} & \hat{f} \\
a & -r_L & 0
\end{pmatrix}
\]
or
\[
\dot{r}_{Lc} = (\dot{x} + r_o \dot{\phi}_c - V \theta) \hat{i}'+ (\dot{y} + a \dot{\phi}_c) \hat{j}'+ 
\]
\[
+ (-a \dot{\phi}_c + r_o \ddot{\phi}_c + V (1 - \frac{r_L}{r_o}) + \dot{z}) \hat{k}'.
\] (A.22)

where \(\frac{\Delta_{\omega}}{a}\) is neglected as small with respect to unity, so that \((a + \Delta_{\omega}) = a\).

Similarly, the velocity of the right wheel contact point is
\[
\dot{r}_{Rc} = (\dot{x} - V \theta + r_{o} \phi) \hat{i}'+ (\dot{y} - a \dot{\phi}) \hat{j}'+ 
\]
\[
+ (\dot{z} + V (1 - \frac{r_{R}}{r_{o}}) - r_{o} \ddot{\phi}_{R} + a \ddot{\phi}) \hat{k}'.
\] (A.23)

The creep forces and moments can now be defined in terms of the velocities that were just derived. Using Equations (A.22) and (A.23), and the axis transformations defined by Equations (A.16) and (A.17), the creep forces and moments are
\[
F_L = F_{Lx} \hat{i}''' + F_{Ly} \hat{j}''' + F_{Lz} \hat{k}'''
\]
\[
\bar{M}_L = M_{Lx} \hat{i}''' + M_{Ly} \hat{j}'''
\]
or (for small displacements),

\[
\mathbf{F}_L = \left[ -\frac{f_{11}}{V} (\dot{x} + r_o \dot{\phi} - V\theta) + \frac{f_{12}}{V} (\delta_L \Omega - \dot{\delta}) \right. \\
\left. - f_{33} \theta \left( 1 - \frac{r_L}{r_o} \right) \right] \hat{i}
\]

\[
\mathbf{M}_L = \left[ \frac{f_{12}}{V} (\dot{x} + r_o \dot{\phi} - V\theta) + \frac{f_{22}}{V} (\delta_L \Omega - \dot{\delta}) \right] \hat{j}
\]

(A.24)

for the left wheel and

\[
\mathbf{F}_R = \mathbf{F}_{R_x} \hat{i} + \mathbf{F}_{R_y} \hat{j} + \mathbf{F}_{R_z} \hat{k}
\]

\[
\mathbf{M}_R = \mathbf{M}_{R_x} \hat{i} + \mathbf{M}_{R_y} \hat{j}
\]

or (for small displacements),

\[
\mathbf{F}_R = \left[ -\frac{f_{11}}{V} (\dot{x} + r_o \dot{\phi} - V\theta) - \frac{f_{12}}{V} (\delta_R \Omega + \dot{\delta}) \right. \\
\left. - f_{33} \theta \left( 1 - \frac{r_R}{r_o} \right) \right] \hat{i}
\]

\[
\mathbf{M}_R = \left[ -\frac{f_{12}}{V} (\dot{x} + r_o \dot{\phi} - V\theta) - \frac{f_{22}}{V} (\delta_R \Omega + \dot{\delta}) \right] \hat{j}
\]

(A.26)

(A.27)

for the right wheel. Note that \( F_{L_y}, F_{R_y}, M_{L_x}, \) and \( M_{R_x} \) are neglected as small because of the products of the creepages with perturbation quantities.

It is reiterated that the creep coefficients are assumed constant and the rails are assumed perfectly rigid in the above derivation of the creep forces and moments.
Suspension Forces and Moments

The suspension forces and moments can be resolved at the wheelset centroid. In general

$$\mathbf{F}_s = F_{s_y} \hat{i}'' + F_{s_x} \hat{j}'' + F_{s_z} \hat{k}''$$

$$= -2 \left( D_{x_p} \ddot{x} + k_{x_p} x \right) \hat{i}'' - (W_{APP} + \Delta F_{s_y}) \hat{j}'' - 2(D_{z_p} \ddot{z} + k_{z_p} z) \hat{k}''$$

(A.28)

where $\Delta F_{s_y}$ is the dynamic component of the vertical loading on the wheelset, and

$$\mathbf{M}_s = M_{s_y} \hat{j}'' + M_{s_z} \hat{k}''$$

$$= -(D_{\theta_p} \ddot{\theta} + k_{\theta_p} \theta) \hat{j}'' + M_{s_z} \hat{k}''$$

(A.29)

where $D_{\theta_p} = 2D_{z_p} a^2$ and $k_{\theta_p} = 2k_{z_p} a^2$.

The flexibility of the axle is modeled as a torsional spring-damper combination that acts about the $\hat{i}'$ axis. Thus,

$$\mathbf{M}_{AX} = M_{AX} \hat{i}' = [D_{AX} (\dot{\beta}_L - \dot{\beta}_R) + k_{AX} (\beta_L - \beta_R)] \hat{i}'$$

(A.30)

Normal Forces

The normal forces acting on each wheel have lateral and vertical components.

For the left wheel,

$$\mathbf{N}_L = N_{L_x} \hat{i}''' + N_{L_y} \hat{j}'''$$

$$= -N_L (\delta_L + \phi) \hat{i}''' + N_L \hat{j}'''$$

(A.31)
For the right wheel,

\[
\overline{N}_R = N_{Rx} \hat{i}'' + N_{Ry} \hat{j}''
\]

\[
= N_R (\delta_R + \phi) \hat{i}'' + N_R \hat{j}'' . \quad \text{(A.32)}
\]

The explicit expressions for the normal forces are obtained in a later section.

**Equations of Motion**

Using the expressions derived in the previous section, the equations of motion for the wheelset are now derived. Newton's second law is applied in all cases.

**Longitudinal Motion**

Applying Newton's second law to motion of the wheelset center of gravity in the longitudinal direction yields

\[
F_{sz} + F_{Lz} + F_{Rz} = m_w \overline{F}_{CG} \cdot \hat{k}''
\]

From the expressions derived in the previous section, this equation becomes

\[
m_w \ddot{z} + \frac{f_{33}}{V} [V(1 - \frac{r_L}{r_0}) + V(1 - \frac{r_R}{r_0})] + 2 \dot{z} - \dot{r}_o (\ddot{\beta}_L + \ddot{\beta}_R) + F_{sz} = 0 . \quad \text{(A.33)}
\]

The torsional motion of the wheelset is antisymmetric, i.e. \( \beta_L = -\beta_R \). Since the wheelset is considered completely symmetric, it is assumed that there is no longitudinal perturbation from the constant velocity of the wheelset center of gravity. Thus,

\[
z = \dot{z} = \ddot{z} = 0 , \text{ and } F_{sz} = 0 .
\]
Equation (A.33) reduces to
\[ r_L + r_R = 2 r_o, \]
or, letting \( r_L = r_o + \Delta r_L \) and \( r_R = r_o + \Delta r_R \), where \( \Delta r_L \) and \( \Delta r_R \)
are the changes in the rolling radii of the left and right wheels, respectively,
\[ \Delta r_L = -\Delta r_R. \]
This is a result of the symmetry of the wheelset, and simply states that the rolling radius of one wheel will decrease by the same amount that the other increases.

**Vertical Motion**

For motion of the wheelset center of gravity in the vertical direction,
\[ -m_w g - (W_{AP} + \Delta F_{g_y}) + N_L + N_R + F_L + F_R = m_w \ddot{r}_{CG} \cdot \ddot{\hat{r}}', \]
or
\[ m_w \ddot{y} + m_w g + W_{AP} + \Delta F_{g_y} - (N_L + N_R) = 0. \]  \hspace{1cm} (A.34)

**Lateral Motion**

The lateral motion of the wheelset center of gravity is given by
\[ N_L x + N_R x + F_s x + F_L x + F_R x = m_w \ddot{r}_{CG} \cdot \ddot{\hat{r}}'', \]
or
\[ m_w \ddot{x} + N_L (\delta + \phi) - N_R (\delta - \phi) + 2 \frac{f_{11}}{V} (\dot{x} + r_o \phi - V \theta) - f_{12} \left( \frac{\delta - \delta_R}{r_o} \right) + 2 \frac{f_{12}}{V} \phi + 2D \ddot{x} + 2k x = 0. \]  \hspace{1cm} (A.35)
Roll Motion

Summing moments about the \( \hat{k}'' \) axis through the wheelset centroid yields

\[
M_{s_z} + \hat{k}''' \cdot [(a+\Delta_L)\hat{i}' - r_L\hat{j}'] \times [(N_L + F_L)\hat{i}''']
+ (N_L + F_L)\hat{j}''' + F_L\hat{k}'''] [-(a-\Delta_R)i' - r_Rj'] \times
[(N_R + F_R)i''' + (N_R + F_R)\hat{j}''' + F_R\hat{k}'''] = \hat{H}_G \cdot \hat{k}'''
\]

or using the expressions derived in the previous section and making small angle assumptions,

\[
(N_L - N_R) a = I w_2 \dot{\phi} - I V w_1 \dot{\theta} + 2 \frac{f_{12} r_0}{V} (\dot{x} r_0 + V \dot{\theta})
+ f_{12} (2 \frac{r_0^2}{V} - (\delta_L - \delta_R) - M_{s_z} \quad (A.36)
\]

Yaw Motion

Summing moments about the \( \hat{j}''' \) axis through the wheelset centroid gives

\[
M_{s_y} + M_{Ry} + M_{Ly} + \hat{j}'''' \cdot [(a+\Delta_L)\hat{i}' - r_L\hat{j}'] \times
[(N_L + F_L)\hat{i}'''' + (N_L + F_L)\hat{j}'''' + F_L\hat{k}'''']
+ [-(a-\Delta_R)i' - r_Rj'] \times [(N_R + F_R)i''' + (N_R + F_R)\hat{j}''' + F_R\hat{k}''']
+ (N_R + F_R)\hat{j}''' + F_R\hat{k}'''] = \hat{H}_G \cdot \hat{j}''''
\]
or

\[-D_{op} \dot{\phi} - k_{op} \dot{\phi} + \frac{2f_{12}}{V} (x + r_0 \dot{\phi} - V\phi)\]

\[+ \frac{2f_{22}}{V} (\delta L - \frac{\delta R}{2} - \Omega - \dot{\phi}) + \frac{af_{33}}{V} [z - r_0 \beta L - a\dot{\phi} + V(1 - \frac{r_1}{r_0})] \]

\[- a\dot{\phi} [z - N_L (\delta L + \phi) - \frac{f_{11}}{V} (x + r_0 \dot{\phi} - V\phi) + \frac{f_{12}}{V} (\delta L \Omega - \dot{\phi})\]

\[- f_{33} \delta (1 - \frac{r_1}{r_0})] - \frac{af_{33}}{V} [z - r_0 \beta R + a\dot{\phi} + V(1 - \frac{r_R}{r_0})] \]

\[- a\dot{\phi} [N_R (\delta R - \phi) - \frac{f_{11}}{V} (x + r_0 \dot{\phi} - V) - \frac{f_{12}}{V} (\delta R \Omega + \dot{\phi})\]

\[- f_{33} \phi (1 - \frac{r_R}{r_0})] = I_{W_2} \ddot{\phi} + I_{W_1} \dot{\phi} \]

After neglecting second order quantities, this lengthy expression reduces to

\[I_{W_2} \ddot{\phi} + I_{W_1} \frac{V}{r_0} \dot{\phi} + D_{op} \dot{\phi} + k_{op} \dot{\phi} - \frac{2f_{12}}{V} (x + r_0 \dot{\phi} - V\phi)\]

\[+ \frac{2f_{22}}{V} [\dot{\phi} - \frac{V}{r_0} (\frac{\delta L - \delta R}{2})] + \frac{2f_{33} a^2}{V} [\dot{\phi} + \frac{V}{r_0} (\frac{r_1 - r_R}{2a})]\]

\[+ \frac{r_0}{a} (\frac{\delta L - \delta R}{2})] = 0 \quad \text{(A.37)} \]
Torsional Motion

Summing moments about the \( \hat{i} \) axis for each wheel gives two equations.

For the left wheel,

\[
-M_{AX} + M_{Lx} - M_{Ly} \phi - r_{L} \hat{j} \times [(N_{Lx} + F_{Lx}) \hat{i}'] + (N_{Ly} + F_{Ly}) \hat{k}' + F_{Lz} \hat{k}' = H_{GL} \hat{i}'
\]

or

\[
\frac{1}{2} I_{wL} \ddot{\beta}_{L} + M_{AX} - M_{Lx} - M_{Ly} \phi + r_{L} \left[ (N_{Lx} + F_{Lx}) \hat{\theta} + F_{Lz} \right] = 0 \quad \text{(A.38)}
\]

Similarly, for the right wheel,

\[
\frac{1}{2} I_{wR} \ddot{\beta}_{R} - M_{AX} - M_{Rx} - M_{Ry} \phi + r_{R} \left[ (N_{Rx} + F_{Rx}) \hat{\theta} + F_{Rz} \right] = 0 \quad \text{(A.39)}
\]

Subtracting Equation (A.39) from Equation (A.38) and substituting the explicit expressions for the forces and moments yields, for small perturbations

\[
\frac{1}{2} I_{wL} (\ddot{\beta}_{L} - \ddot{\beta}_{R}) + 2 D_{AX} (\dddot{\beta}_{L} - \dddot{\beta}_{R}) + 2 k_{AX} (\dot{\beta}_{L} - \dot{\beta}_{R}) + f_{33} (r_{L} - r_{R}) + 2 \frac{f_{33}}{V} r_{0} a \delta + 2 \frac{f_{33}}{V} r_{0}^{2} (\dot{\beta}_{L} - \dot{\beta}_{R}) = 0
\]

or, since \( \beta_{L} = -\beta_{R} = \beta/2 \),

\[
\frac{1}{2} I_{wL} \dot{\beta} + \left( 2 D_{AX} \frac{r_{L} - r_{R}}{V} \right) \ddot{\beta} + 2 k_{AX} \beta' + 2 \frac{f_{33}}{V} r_{0}^{2} a \delta + 2 f_{33} a \frac{r_{L} - r_{R}}{2a} = 0 \quad \text{(A.40)}
\]

where \( D_{AX} \) is assumed to be a viscous damper applied to the axle.

(The structural damping of the axle is assumed to be negligible).
The axle torsional stiffness, $k_{AX}$, is obtained from the following formula for the torsional flexibility of homogeneous circular shafts:

$$k_{AX} = \frac{\pi d^4 G}{32 \xi_o}.$$  \hfill (A.41)

Thus, changing the axle stiffness may be accomplished by either changing the axle diameter, length or material properties. As mentioned previously, the torsional stiffness and damping of the wheelset may be provided by resilient wheels or other unconventional designs. Obviously Equation (A.41) would not be valid for determining the torsional stiffness due to any mechanism other than from the flexibility of the axle.

**Normal Forces**

As the equation for longitudinal motion of the wheelset was reduced to a kinematic relation, the motion of the wheelset is described by five equations in the five unknowns $x$, $\theta$, $\beta$, $N_L$, and $N_R$. Thus, the explicit expressions for the normal forces can be obtained.

Multiplying Equation (A.34) by "a" and rearranging gives

$$ (N_L + N_R)a = (m \dddot{y} + W'_{APP} + m_w g + \Delta F_{sy})a $$

Adding this to Equation (A.36) results in the expression

$$ N_L = \frac{1}{2a} \{ (m \dddot{y} + W'_{APP} + \Delta F_{sy} + m_w g) a + \left[ I_{w_2} \dddot{\phi}_y - I_{w_1} \frac{V}{r_o} \dot{\theta} + 2 \frac{f_{11}r_o}{V}(x + r_o \dot{\phi} - V\theta) \right] $$

$$ + 2f_{12} \left( \frac{r_o}{V} \dot{\theta} - (\dot{\varepsilon}_L - \dot{\varepsilon}_R)/2 - M_{sz} \right) \} \hfill (A.42) $$
Subtracting the two equations yields
\[
N_R = \frac{1}{2a} \left\{ \left( m_w \dddot{x} + W_{\text{App}} + \Delta F_{s_y} + m_w g \right) - \left[ I_{w2} \dddot{\phi} - I_{w1} \frac{V}{r_0} \dddot{\phi} + 2f_{11}r_0 (\dddot{x} + r_0 \dddot{\phi} - V \theta) / V \right. \\
+ \left. 2f_{12} \left( -\frac{0}{V} \dddot{\phi} - \left( \delta_L - \delta_R \right) \right) - M_{s_z} \right\},
\]
(A.43)

By using Equations (A.42) and (A.43), the first two terms in the lateral equation (Equation (A.35)) may be defined explicitly. For small perturbations
\[
(N_L + N_R) \dot{\phi} = W_{\text{App}} \dot{\phi}
\]
(A.44)

and
\[
N_R \delta_R - N_L \delta_L = \left( \frac{\delta_R - \delta_L}{2z} \right) W_{\text{App}} + M_{s_z} \frac{\delta_L + \delta_R}{2a} + I_{w1} \frac{V}{r_0 a} \delta_0 \delta
\]
(A.45)

where products of \( \Delta F_{s_y} \), \( \delta_L \), \( \delta_R \) and \( \phi \) and products of \( M_{s} \) and \( \phi \) are neglected as small, and \( W_{\text{App}} = m_w g + W_{\text{App}}' \).

The term \( I_{w1} V \delta_0 \delta / (r_0 a) \) is retained in Equation (A.45) from the dynamic effects of \( N_L \) and \( N_R \). The other dynamic terms in Equation (A.45) will be small in comparison to other terms in the lateral force equation.

A detailed discussion of the derivation of the expressions for the normal forces is presented at the end of this section.

Using Equations (A.44) and (A.45), the wheelset lateral equation becomes
\[
m_w \dddot{x} + 2f_{11} (\dddot{x} + r_0 \dddot{\phi} - V \theta) + 2f_{12} \left( -\frac{0}{V} \dddot{\phi} - \left( \delta_L - \delta_R \right) \right) \frac{\delta_0 \delta}{2z} - W_{\text{App}} \frac{\delta_L - \delta_R}{2} - I_{w1} \frac{V}{r_0 a} \delta_0 \delta - M_{s_z} \frac{\delta_L + \delta_R}{2a} + 2D \dot{x} + 2k \dddot{x} = 0
\]
(A.46)

where the term \( W_{\text{App}} (\dddot{x} + \delta_L - \delta_R / 2) \) is the gravitational stiffness force that is discussed in Chapter II. The gravitational stiffness force is the net lateral component of the normal forces, and usually
has a stabilizing effect on wheelset motions.

Equations (A.37), (A.40), and (A.46) fully describe the motion of a general wheelset for small oscillations about its equilibrium position.

Linearization of Kinematic Quantities

For small perturbations and rigid rails: the dependence on the wheelset yaw and spin motions of roll angle, rolling radii and contact angles can be neglected to a good approximation. These kinematic quantities are then functions of the lateral motion only, and a MacLaurin series can be written for the terms in the equations of motion which contain the quantities.

Thus, neglecting quantities of higher order than one in the perturbation quantities, the kinematic quantities may be approximated as

\[ \phi = \phi_0 + a_1 \frac{x}{a} \]
\[ \frac{r_L - r_R}{2a} = \rho_0 + \lambda_1 \frac{x}{a} \]
\[ \frac{\delta_L - \delta_R}{2} = \Delta_0 + \Delta_1 \frac{x}{a} \]
\[ \frac{\delta_L + \delta_R}{2} = \delta_0 + \delta_1 \frac{x}{a} \] (A.47)

For new wheels on new rails, the coefficients of Equation (A.47) are approximately constant for all amplitudes of wheelset lateral displacements less than those corresponding to flange contact [15].
For curved or worn wheel and rail profiles, the coefficients may be functions of the wheelset lateral displacements. This functional dependence may be obtained by one of several methods.

If the wheelset lateral displacement is assumed to be sinusoidal, sinusoidal describing functions may be obtained for the kinematic quantities that appear on the left-hand sides of Equations (A.47) (See Figure 2.2). Alternatively, if a nonlinear function, \( y = F(x) \) is approximated by \( G(x)\alpha \), then \( G(x_0) \) is the slope of the line from the origin to the curve at \( x = x_0 \). This method has been used by British Rail in defining the effective conicity.

A third method is to define a desired value of the amplitude, \( x = x_o \). Then, \( G(x_0) \) can be defined by fitting a straight line in the least-squares sense to the curve over \( 0 < x < x_0 \).

Assuming symmetric wheels and rails, the non-zero terms of Equations (A.47) are \( a_1, \lambda_1, \Delta_1, \) and \( \delta_0 \). Retaining only the non-zero terms of Equations (A.47), the three equations of motion for the vehicle model finally become (after dropping \( r_o a_1/a \) as small compared to unity)

\[
\begin{align*}
\mathbf{m} \ddot{x} + 2(D_{x_p} + f_{11}/V)\dot{x} + [2k_{x_p} + W_{App}(\Delta_1+a_1)]/a \\
-2f_{12}\Delta_1/(r_o a)\alpha + [(2f_{12}/V) - I_{w_1} V\delta_0/(r_o a)]\delta \\
- 2f_{11}\theta - M_z \delta_0/a = 0 \quad \text{(Wheelset Lateral Equation)} \quad (A.48)
\end{align*}
\]

\[
\begin{align*}
I_{w_2} \ddot{\theta} + [D_{\theta_p} + 2(f_{33} r_o^2 + f_{22})/V]\theta + (k_{\theta_p} + 2f_{12})\theta \\
+ [I_{w_1} V a_1/(r_o a) - 2f_{12}/V]\dot{x} + 2[f_{33} a\lambda_1/r_o \\
- f_{22}\Delta_1/(r_o a)]\alpha + (f_{33} r_o a/V)\delta = 0 \quad \text{(Wheelset Yaw Equation)} \quad (A.49)
\end{align*}
\]
As explained below, the term $M_{s2}\delta_1/a$ may be neglected.

Development of the Gravitational Stiffness Force

The gravitational stiffness force (G.S.F.) is the lateral resultant of the normal forces of the rails on the wheels:

$$G.S.F. = N_L \sin(\delta_1 + \phi) - N_R \sin(\delta_R - \phi)$$

$$\approx (N_L + N_R)\phi + N_L \delta_L - N_R \delta_R$$  \hspace{1cm} (A.51)

Using Equations (A.44) and (A.45)

$$G.S.F. \approx W_{\text{App}}[(\delta_L - \delta_R)/2 + \phi) - (M_{s2}/a)(\delta_L + \delta_R)/2]$$  \hspace{1cm} (A.52)

In the above expression, the term $(M_{s2}/a)(\delta_L + \delta_R)/2$ is the effect on wheelset lateral motions of the roll moment, which is exerted by the rest of the vehicle about the wheelset centroid. It consists of the offset of the load applied by the vehicle and any dynamic loadings that arise from motions of the vehicle. For this model, the dynamic component of the roll moment, when multiplied by $\delta_L + \delta_R/2a$, is a second order quantity and is neglected.

Consider a static configuration of the wheelset model with knife-edge rails, as sketched below. The roll moment is due
to the offset of the applied load, $W_{\text{App}}$, from the equilibrium position.

Wheelset on Knife-Edge Rails (Front View).

For knife-edge rails, the contact point on the rails remains stationary (This is an accurate approximation for new wheels on new rails). Thus, summing moments about the contact point on each wheel results in the following expressions:

\[
N_L = \frac{1}{2a \cos(\delta_L + \phi)} \left[ m_w g (a+x) + W_{\text{App}}' (a+x') \right] \quad (A.53a)
\]

\[
N_R = \frac{1}{2a \cos(\delta_R - \phi)} \left[ m_w g (a-x) + W_{\text{App}}' (a-x') \right] \quad (A.53b)
\]

Then, the G.S.F. is

\[
N_L \sin(\delta_L + \phi) - N_R \sin(\delta_R - \phi) = \\
\frac{1}{2} \left[ m_w g \left(1+\frac{x}{a}\right) + W_{\text{App}}' \left(1+x'\right) \right] \tan(\delta_L + \phi)
\]

\[
- \frac{1}{2} \left[ m_w g \left(1-\frac{x}{a}\right) + W_{\text{App}}' \left(1-x'\right) \right] \tan(\delta_R - \phi) \quad (A.54)
\]

For small $(\delta_L + \phi)$ and small $(\delta_R - \phi)$, this expression reduces to

\[
\text{G.S.F.} = W_{\text{App}}' \left[ (\delta_L - \delta_R)/2 + \phi \right] \left[ (W_{\text{App}}x'/a) + (m_w g x/a) \right] (\delta_L + \delta_R)/2. \quad (A.55)
\]
It should be re-emphasized that an assumption used to arrive at Equation (A.55) is that the contact point on the rail remains stationary. This may be valid for new wheels on new rails for contact in the tread region. For worn wheels and/or rails the locations of the points of contact on the rails may change with wheelset lateral displacement. As an example, the location of the wheel/rail contact points as a function of wheelset lateral displacement for new wheels on new rails and for worn wheels on worn rails are compared in Figure A.5. The data shown in this figure were obtained by experimental and analytical techniques by Cooper-rider et al [15].

Consider next the general case for a nonstationary contact point on the rail. The static configuration is sketched below.

Wheelset on Profiled Rails (Front View).
FIGURE A.5 Location of Wheel and Rail Contact Points for Varying Wheelset Lateral Displacement (from [15]).
The terms $\Delta_{lr}$ and $\Delta_{rr}$ are the displacements of the contact points on the left and right rails, respectively. Then, the following expressions are obtained by taking moments about each contact point:

$$N_L = \frac{1}{1 - \frac{\Delta_{rl}^2 + \Delta_{r2}^2}{2a}} \times \left\{ [1 - (\Delta_{rl}/a)](m_wg + W'_{App})/2 \right. \\
+ \left. [(m_wg/a) + (W'_{Appx'/a})]/2 \right\}$$

(A.56a)

$$N_R = \frac{1}{1 - \frac{\Delta_{rl}^2 + \Delta_{r2}^2}{2a}} \times \left\{ [1 - (\Delta_{rl}/a)](m_wg + W'_{App})/2 \right. \\
- \left. [(m_wg/a) + (W'_{Appx'/a})]/2 \right\}.$$  

(A.56b)

Then, the G.S.F. is, for small $(\delta_L + \phi)$ and $(\delta_R - \phi)$

$$\text{G.S.F.} = \frac{1}{1 - \frac{\Delta_{rl}^2 + \Delta_{r2}^2}{2a}} \times \left\{ (W'_{App}) \left( 1 - \frac{\Delta_{rl} + \Delta_{r2}}{2a} \right) \phi \right. \\
+ \left. \frac{\delta_L}{2} [1 - (\Delta_{rl}/a)] - \frac{\delta_R}{2} [1 - (\Delta_{rl}/a)] \right\} \\
+ \left. \left[ (\delta_L + \delta_R)/2 \right] (m_wg/a) + (W'_{Appx'/a}) \right\}.$$  

(A.57)
The terms $\Delta_{kr}$ and $\Delta_{rr}$ are kinematic quantities, i.e., they describe the wheel and rail contact geometry. For small displacements of the wheelset, $\Delta_{kr}$ and $\Delta_{rr}$ may be considered as nonlinear functions of the wheelset lateral displacement only. Wheelset motions other than the lateral displacement have higher order effect on the terms $\Delta_{kr}$ and $\Delta_{rr}$. Thus, $\Delta_{kr}$ and $\Delta_{rr}$ may be treated like the other kinematic quantities, i.e., expanded in a Taylor series in $x$. Then, the coefficients of the expansion may be determined by one of several linear techniques (e.g., describing functions and least-squares fit).

Since $\Delta_{kr}/a$ and $\Delta_{rr}/a$ are small with respect to unity, Equation (A.57) is reduced to Equation (A.55). Thus, the effect on gravitational stiffness force of the shift in the contact point on the rails is negligible. For knife-edge rails and for new wheels on new rails, $\Delta_{kr} = \Delta_{rr} = 0$ for contact in the tread region. For the particular worn wheel and rail profiles shown in Figure A.5, $\Delta_{kr}$ and $\Delta_{rr}$ are also approximately zero for small wheelset motions from equilibrium. However, this is not true in general for curved wheel profiles.

Comparing Equations (A.52) and (A.55), it may be seen that the roll moment applied to the wheelset may be represented as a lateral weight shift (neglecting dynamic effects in $M_{Sz}$). When the reference vehicle translates uniformly along the track, $x'=0$. If we also neglect $m_wg x (\delta_L + \delta_R)/2a$ as small, the gravitational stiffness force is given by

$$G.S.F. = W_{App} \left[ (\delta_L - \delta_R)/2 + \phi \right].$$  
(A.58)
This is the representation used in the analyses conducted during this investigation.
### TABLE A.1 MATRIX ELEMENTS FOR WHEELSET MODEL

\[ \sum_{j=1}^{3} A_{ij} x_j = 0 \quad i = 1, 2, 3. \]

where

\[ A_{ij} = s^2 m_{ij} + s c_{ij} + k_{ij} \]

<table>
<thead>
<tr>
<th>Row</th>
<th>Column</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Wheelset Lateral Equation</td>
<td>( x )</td>
</tr>
<tr>
<td>2. Wheelset Yaw Equation</td>
<td>( \theta )</td>
</tr>
<tr>
<td>3. Wheelset Torsional Equation</td>
<td>( \beta )</td>
</tr>
</tbody>
</table>

#### ELEMENTS OF MASS MATRIX

- \( m_{11} = m_w \)
- \( m_{22} = I_{w_2} \)
- \( m_{33} = I_{w_1}/2 \)

#### ELEMENTS OF DAMPING MATRIX

- \( c_{11} = 2f_{11}/V + D_x p \)
- \( c_{12} = 2f_{12}/V + I_{w_1}\delta_o V/r_o a \)
- \( c_{21} = I_{w_1} V a_1/(r_o a) - 2f_{12}/V \)
- \( c_{22} = 2(f_{22} + f_{33} a^2)/V + D_{p} \)
- \( c_{23} = f_{33} r_o a/V \)
- \( c_{32} = 2f_{33} r_o a/V \)
- \( c_{33} = 2D_{A} x + f_{33} r_o^2 /V \)
TABLE A.1 (Continued)

ELEMENTS OF STIFFNESS MATRIX*

\[ k_{11} = W_{pp}(a_1 + \Delta_1)/a \]
\[ - 2f_{11} \Delta_1 / r o + 2k_{1p} \]

\[ k_{12} = -2f_{11} \]

\[ k_{21} = 2f_{33} \lambda_1 / r o - 2f_{22} \Delta_1 / r o \]

\[ k_{22} = 2f_{12} + k_{2p} \]

\[ k_{31} = 2f_{33} \lambda_1 \]

\[ k_{33} = 2k_{AX} \]

*Note that for this model, the term \( M_s \frac{\delta_L + \delta_R}{2a} \) is neglected as small.

Indeed, from Equation (A.52) with \( \Delta_{kr} \) and \( \Delta_{rr} \) small with respect to \( a \),

\[ M_s (\delta_L + \delta_R)/2a = (W_{pp} x'/a + m_w g x/a)(\delta_L + \delta_R)/2 \]

if dynamic contributions are neglected as small. The value of \( x' \) is zero and \( m_w g x(\delta_L + \delta_R)/2a \) is neglected as small.
B. EQUATIONS OF MOTION FOR 11 DEGREE OF FREEDOM MODEL

Schematics of the 11 DOF model are shown in Figures 4.1, 4.2, and 4.3. The model is comprised of a 9 DOF generalized truck model and a 2 DOF car body. The truck model has lateral, yaw, and torsional degrees of freedom for each of two wheelsets and lateral, yaw, and warp degrees of freedom for the truck frame. The possibility of wheelset interconnections is included. The "pseudo-car body" may move laterally and in roll.

The equations of motion for the 11 DOF model are listed in matrix form in Table B.1.
TABLE B.1 MATRIX ELEMENTS FOR 11 DOF MODEL

\[ \sum_{j=1}^{3} A_{ij} x_j = 0 \quad i = 1, 2, 3, \ldots, 11 \]

where

\[ A_{ij} = s m_{ij} + s c_{ij} + k_{ij} \]

<table>
<thead>
<tr>
<th>Row</th>
<th>Column</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Front Wheelset Lateral Equation</td>
<td>( x_{w_1} )</td>
</tr>
<tr>
<td>2. Front Wheelset Yaw Equation</td>
<td>( \theta_{w_1} )</td>
</tr>
<tr>
<td>3. Front Wheelset Torsional Equation</td>
<td>( \beta_1 )</td>
</tr>
<tr>
<td>4. Rear Wheelset Lateral Equation</td>
<td>( x_{w_2} )</td>
</tr>
<tr>
<td>5. Rear Wheelset Yaw Equation</td>
<td>( \theta_{w_2} )</td>
</tr>
<tr>
<td>6. Rear Wheelset Torsional Equation</td>
<td>( \beta_2 )</td>
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<tr>
<td>7. Truck Lateral Equation</td>
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<tr>
<td>8. Truck Yaw Equation</td>
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<tr>
<td>9. Truck Warp Equation</td>
<td>( \theta_W )</td>
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<tr>
<td>10. Car Body Lateral Equation</td>
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</tr>
<tr>
<td>11. Car Body Roll Equation</td>
<td>( \phi_C )</td>
</tr>
</tbody>
</table>

ELEMENTS OF MASS MATRIX

\[
\begin{align*}
  m_{11} &= m_w \\
  m_{18} &= m_{w^2} \\
  m_{28} &= I_{w_2} \\
  m_{33} &= \frac{I_{w_1}}{2} \\
  m_{17} &= m_w \\
  m_{22} &= I_{w_2} \\
  m_{29} &= I_{w_2} \\
  m_{44} &= m_w 
\end{align*}
\]
### TABLE B.1 (Cont'd)

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<td>$m_{59}$</td>
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<td>$m_{98}$</td>
<td>$I_{w_2} + 2m_sd^2$</td>
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<td>$m_{10}$</td>
<td>$m_Bh$</td>
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<tr>
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**ELEMENTS OF DAMPING MATRIX**

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<td>$2(f_{11}/V)$</td>
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<td>$2f_{12}/V - I_{w_1}V\delta_o/r_oa$</td>
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<td>$c_{22}$</td>
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<td>$c_{23}$</td>
<td>$f_{33}r_oa/V$</td>
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<td>$c_{21}$</td>
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### TABLE B.1 (Cont'd)

**ELEMENTS OF STIFFNESS MATRIX**

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<tr>
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<td>$W_{app} \Delta_1 + a_1/a - 2f_{12} \Delta_1/r_o$ + $2k_{x_p}^2 + k_{x_L}^2$</td>
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<tr>
<td>$k_{12}$</td>
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<tr>
<td>$k_{28}$</td>
<td>$2f_{12} + 2(f_{33} a \Delta_1 - f_{22} \Delta_1/a)(\ell/r_o)$</td>
</tr>
<tr>
<td>$k_{29}$</td>
<td>$2f_{12} + 2k_{x_L}^2$</td>
</tr>
<tr>
<td>$k_{31}$</td>
<td>$2f_{33} a \Delta_1$</td>
</tr>
<tr>
<td>$k_{33}$</td>
<td>$2k_{AX}$</td>
</tr>
<tr>
<td>$k_{37}$</td>
<td>$2f_{33} a \Delta_1$</td>
</tr>
<tr>
<td>$k_{38}$</td>
<td>$2f_{33} a \Delta_1 \ell$</td>
</tr>
<tr>
<td>$k_{41}$</td>
<td>$-k_{x_L}$</td>
</tr>
<tr>
<td>$k_{42}$</td>
<td>$k_{x_L}^2$</td>
</tr>
</tbody>
</table>
TABLE B.1 (Cont'd)

\[ k_{44} = W_{\text{App}}(\Delta_1 + a_1)/a - 2f_{12}\Delta_1/r_o a + 2k_x p + k_x l \]
\[ k_{45} = -2f_{11} + k_x l \]
\[ k_{47} = W_{\text{App}}(\Delta_1 + a_1)/a - 2f_{12}\Delta_1/r_o a \]
\[ k_{48} = -[W_{\text{App}}(a_1 + \Delta_1)/a - 2f_{12}\Delta_1/r_o a] l - 2f_{11} \]
\[ k_{49} = -2f_{11} + 2k_x l \]
\[ k_{51} = -k_x l \]
\[ k_{52} = -k_{\theta L} + k_x l^2 \]
\[ k_{54} = 2f_{33}\alpha_1/r_o \]
\[ k_{55} = 2(f_{12} + k_z d^2) + k_{\theta L} + k_x l^2 \]
\[ k_{56} = 2f_{12} + 2k_x l^2 \]
\[ k_{57} = 2f_{33}\alpha_1/r_o - 2f_{22}\Delta_1/r_o a \]
\[ k_{58} = 2f_{12} - (2f_{33}\alpha_1/r_o - 2f_{22}\Delta_1/r_o a) l \]
\[ k_{59} = 2f_{12} + 2k_x l^2 \]
TABLE B.1 (Cont'd)

<table>
<thead>
<tr>
<th>k</th>
<th>66</th>
<th>k = 2f \lambda</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td></td>
<td>f x</td>
</tr>
<tr>
<td>67</td>
<td></td>
<td>2f \lambda</td>
</tr>
<tr>
<td>71</td>
<td></td>
<td>-2k_x x_p</td>
</tr>
<tr>
<td>77</td>
<td></td>
<td>2k x</td>
</tr>
<tr>
<td>82</td>
<td></td>
<td>-2k_z p d^2</td>
</tr>
<tr>
<td>85</td>
<td></td>
<td>-2k_z p d^2</td>
</tr>
<tr>
<td>95</td>
<td></td>
<td>-2k_z p d^2</td>
</tr>
<tr>
<td>99</td>
<td></td>
<td>k_\theta_{CP} + k_\theta_W</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>2k_x</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>2k_x</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>-k_y d^2 a_1 / a</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>-2k_x h - 2k_y d^2 a_1 / a</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>2k_x h^2 + 2k_y d^2 - m_c g h</td>
</tr>
</tbody>
</table>

k = 2k_{AX}

k = 2f \lambda_k

k = -2k_{x_p}

k = -2k_x

k = -2k_{x_p}

k = 2k_x

k = k_{\theta_{CP}}

k = -2k_{z_p} d^2

k = k_{\theta_{CP}}

k = -2k_x

k = 2k_x

k = 2k_{x_p}

k = -k_y d^2 a_1 / a

k = -2k_x h - 2k_y d^2 a_1 / a

k = -k_y d^2 a_1 / a

k = 2k_x

k = 2k_{x_p}
C. EQUATIONS OF MOTION FOR 23 DOF, COMPLETE VEHICLE MODEL

Schematics of the 23 DOF model are shown in Figures 5.1 and 5.2. The model is comprised of two 9 DOF generalized truck models (modified to the complete car case) and a 5 DOF car body having three degrees of freedom corresponding to rigid body lateral, yaw, and roll and an additional two degrees of freedom corresponding to an approximation of the first lateral bending and first torsional modes.

The equations of the front and rear trucks are essentially the same as those for the 11 DOF model. Modifications are made to include the moments due to the fixed ends of the secondary spring groups. However, it is expected that the moments due to the fixed ends are much less than that due to either the vertical or lateral deflection of the springs times the appropriate moment arm in terms where both effects are present.

The equations of motion for the complete vehicle model are listed in matrix form in Table C.1.
TABLE C.1  MATRIX ELEMENTS FOR COMPLETE  
VEHICLE MODEL

\[ \sum_{j=1}^{23} A_{ij} x_j = 0 \quad i = 1, 3, \ldots, 23 \]

where:

\[ A_{ij} = s_{ij} + c_{ij} + k_{ij} \]

<table>
<thead>
<tr>
<th>ROW</th>
<th>COLUMN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Front Truck Front Wheelset Lateral Equation</td>
<td>( x_{W_1} )</td>
</tr>
<tr>
<td>2. Front Truck Front Wheelset Yaw Equation</td>
<td>( \theta_{W_1} )</td>
</tr>
<tr>
<td>3. Front Truck Front Wheelset Torsional Equation</td>
<td>( \beta_{W_1} )</td>
</tr>
<tr>
<td>4. Front Truck Rear Wheelset Lateral Equation</td>
<td>( x_{W_2} )</td>
</tr>
<tr>
<td>5. Front Truck Rear Wheelset Yaw Equation</td>
<td>( \theta_{W_2} )</td>
</tr>
<tr>
<td>6. Front Truck Rear Wheelset Torsional Equation</td>
<td>( \beta_{W_2} )</td>
</tr>
<tr>
<td>7. Front Truck Lateral Equation</td>
<td>( x_{TF} )</td>
</tr>
<tr>
<td>8. Front Truck Yaw Equation</td>
<td>( \theta_{TF} )</td>
</tr>
<tr>
<td>9. Front Truck Warp Equation</td>
<td>( \theta_{WF} )</td>
</tr>
<tr>
<td>10. Rear Truck Front Wheelset Lateral Equation</td>
<td>( x_{W_3} )</td>
</tr>
<tr>
<td>11. Rear Truck Front Wheelset Yaw Equation</td>
<td>( \theta_{W_3} )</td>
</tr>
<tr>
<td>12. Rear Truck Front Wheelset Torsional Equation</td>
<td>( \beta_{W_3} )</td>
</tr>
<tr>
<td>13. Rear Truck Rear Wheelset Lateral Equation</td>
<td>( x_{W_4} )</td>
</tr>
<tr>
<td>14. Rear Truck Rear Wheelset Yaw Equation</td>
<td>( \theta_{W_4} )</td>
</tr>
<tr>
<td>15. Rear Truck Rear Wheelset Torsional Equation</td>
<td>( \beta_{W_4} )</td>
</tr>
<tr>
<td>16. Rear Truck Lateral Equation</td>
<td>( x_{TR} )</td>
</tr>
<tr>
<td>17. Rear Truck Yaw Equation</td>
<td>( \theta_{TR} )</td>
</tr>
<tr>
<td>18. Rear Truck Warp Equation</td>
<td>( \theta_{WR} )</td>
</tr>
<tr>
<td>19. Car Body Rigid Lateral Equation</td>
<td>( x_{RIG} )</td>
</tr>
<tr>
<td>20. Car Body Rigid Yaw Equation</td>
<td>( \theta_{RIG} )</td>
</tr>
<tr>
<td>21. Car Body Flexible Lateral Equation</td>
<td>( \theta_{FLEX} )</td>
</tr>
<tr>
<td>22. Front Half-Car Body Roll Equation</td>
<td>( \phi_{CF} )</td>
</tr>
<tr>
<td>23. Rear Half-Car Body Roll Equation</td>
<td>( \phi_{CR} )</td>
</tr>
</tbody>
</table>
### TABLE C.1 (Cont'd)

**Elements of Mass Matrix**

<table>
<thead>
<tr>
<th>Element</th>
<th>Expression</th>
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</thead>
<tbody>
<tr>
<td>$m_{11}$</td>
<td>$m_w$</td>
</tr>
<tr>
<td>$m_{18}$</td>
<td>$lm_w$</td>
</tr>
<tr>
<td>$m_{28}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{33}$</td>
<td>$I_{w_1}/2$</td>
</tr>
<tr>
<td>$m_{47}$</td>
<td>$m_w$</td>
</tr>
<tr>
<td>$m_{55}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{59}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{77}$</td>
<td>$2m_s$</td>
</tr>
<tr>
<td>$m_{89}$</td>
<td>$I_{B_2} + 2m_s d^2$</td>
</tr>
<tr>
<td>$m_{99}$</td>
<td>$I_{B_2} + 2m_s d^2$</td>
</tr>
<tr>
<td>$m_{10;16}$</td>
<td>$m_w$</td>
</tr>
<tr>
<td>$m_{11;11}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{11;18}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{17}$</td>
<td>$m_w$</td>
</tr>
<tr>
<td>$m_{22}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{29}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{44}$</td>
<td>$m_w$</td>
</tr>
<tr>
<td>$m_{48}$</td>
<td>$-lm_w$</td>
</tr>
<tr>
<td>$m_{58}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{66}$</td>
<td>$I_{w_1}/2$</td>
</tr>
<tr>
<td>$m_{88}$</td>
<td>$2(I_{s_2} + m_s d^2) + I_{B_2}$</td>
</tr>
<tr>
<td>$m_{98}$</td>
<td>$I_{B_2} + 2m_s d^2$</td>
</tr>
<tr>
<td>$m_{10;17}$</td>
<td>$2m_w$</td>
</tr>
<tr>
<td>$m_{11;17}$</td>
<td>$I_{w_2}$</td>
</tr>
<tr>
<td>$m_{12;12}$</td>
<td>$I_{w_1}/2$</td>
</tr>
</tbody>
</table>
TABLE C.1 (Cont'd)

\[ m_{13}\ 13 = m_w \]
\[ m_{13}\ 17 = -2m_w \]
\[ m_{14}\ 17 = I_{w_2} \]
\[ m_{15}\ 15 = I_{w_1}/2 \]
\[ m_{17}\ 17 = 2(I_{s_2} + m_s d^2) + I_{B_2} \]
\[ m_{18}\ 17 = I_{B_2} + 2m_s d^2 \]
\[ m_{19}\ 19 = m_c + 2m_B \]
\[ m_{19}\ 22 = h m_B + (h-h_{CG})m_c/2 \]
\[ m_{20}\ 20 = \frac{1}{2}m_B(L+2b_o)^2 + I_{c_2} \]
\[ m_{20}\ 23 = -hm_B[(L/2) + b_o] \]
\[ -m_c L(h - h_{CG})/4 \]
\[ m_{21}\ 21 = (I_{c_2} - m_c L^2/4) \]
\[ m_{21}\ 23 = -hb_o m_B \]
\[ m_{22}\ 20 = hm_B[(L/2) + b_o] \]
\[ - (m_c/2)(L/2)(h_{CG}-h) \]
\[ m_{22}\ 22 = I_{B_3} + m_B h^2 + (I_{c_3}/2) + m_c (h_{CG}-h)^2/2 \]
\[ m_{13}\ 16 = m_w \]
\[ m_{14}\ 14 = I_{w_2} \]
\[ m_{14}\ 18 = I_{w_2} \]
\[ m_{16}\ 16 = 2m_s \]
\[ m_{17}\ 18 = I_{B_2} + 2m_s d^2 \]
\[ m_{18}\ 18 = I_{B_2} + 2m_s d^2 \]
\[ m_{19}\ 21 = b_0 m_c \]
\[ m_{19}\ 23 = h m_B + (h-h_{CG})m_c/2 \]
\[ m_{20}\ 22 = hm_B[(L/2) + b_o] \]
\[ + m_c L(h - h_{CG})/4 \]
\[ m_{21}\ 19 = -m_b b_o \]
\[ m_{21}\ 22 = -hb_o m_B \]
\[ m_{22}\ 19 = h m_B - (m_c/2)(h_{CG}-h) \]
\[ m_{22}\ 21 = -(m_c/2)(h_{CG}-h)b_o \]
\[ m_{23\ 19} = h m_B - \left(\frac{m_c}{2}\right)(h_{CG} - h) \]

\[ m_{23\ 21} = -\left(\frac{m_c}{2}\right)(h_{CG} - h)b_o \]

**ELEMENTS OF DAMPING MATRIX**

\[ c_{11} = 2D_{x_p} + 2f_{11}/V \]

\[ c_{17} = 2f_{11}/V \]

\[ c_{19} = \left(2f_{12}/V\right) - I_{w_1} V_s o / (r_o a) \]

\[ c_{22} = 2(f_{33a^2}/V + f_{22}/V + D_{z_p} d^2) \]

\[ c_{27} = (a_1/a)(I_{w_1} V/r_o) - 2f_{12}/V \]

\[ c_{29} = 2(a^2f_{33} + f_{22})/V \]

\[ c_{33} = 2D_{A_X} + r_o^2f_{33}/V \]

\[ c_{39} = 2a_o f_{33}/V \]

\[ c_{45} = \left(2f_{12}/V\right) - I_{w_1} V_s o / (a r_o) \]

\[ m_{23\ 20} = -h m_B[(L/2) + b_o] + \left(\frac{m_c}{2}\right)(h_{CG} - h)(L/2) \]

\[ m_{23\ 23} = I_{B_3} + h^2 m_B + (I_{c_3}/2) + m_c(h_{CG} - h)^2/2 \]

\[ c_{12} = \left(2f_{12}/V\right) - I_{w_1} V_s o / (r_o a) \]

\[ c_{18} = 2f_{11}/V \]

\[ c_{21} = I_{w_1} V_o / (r_o a) - 2f_{12}/V \]

\[ c_{23} = a r_o f_{33}/V \]

\[ c_{28} = [I_{w_1} V_o / (r_o a)] \] + \[2(f_{33a^2} + f_{22})/V \]

\[ c_{32} = 2a_o f_{33}/V \]

\[ c_{38} = 2a_o f_{33}/V \]

\[ c_{44} = 2D_{x_p} + 2f_{11}/V \]

\[ c_{47} = 2f_{11}/V \]
TABLE C.1 (Cont'd)

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{48}$</td>
<td>$-2\alpha f_{11}/V$</td>
</tr>
<tr>
<td></td>
<td>$+(2f_{12}/V)-I_{W_1}V\delta_o/(a_r O)$</td>
</tr>
<tr>
<td>$c_{54}$</td>
<td>$I_{W_1}V a_1/(r_o a) - 2f_{12}/V$</td>
</tr>
<tr>
<td>$c_{56}$</td>
<td>$a_r O f_{33}/V$</td>
</tr>
<tr>
<td>$c_{58}$</td>
<td>$[2(a^2 f_{33}+f_{22})/V]$</td>
</tr>
<tr>
<td></td>
<td>$-\lambda[I_{W_1}V a_1/(r_o a) - 2f_{12}/V]$</td>
</tr>
<tr>
<td>$c_{65}$</td>
<td>$2a_r O f_{33}/V$</td>
</tr>
<tr>
<td>$c_{68}$</td>
<td>$2a_r O f_{33}/V$</td>
</tr>
<tr>
<td>$c_{71}$</td>
<td>$-2D_x P$</td>
</tr>
<tr>
<td>$c_{77}$</td>
<td>$2D_x$</td>
</tr>
<tr>
<td>$c_{79}$</td>
<td>$-2D_x$</td>
</tr>
<tr>
<td>$c_{72}$</td>
<td>$-2hD_x$</td>
</tr>
<tr>
<td>$c_{82}$</td>
<td>$-2D_z d^2$</td>
</tr>
<tr>
<td>$c_{85}$</td>
<td>$-2D_z d^2$</td>
</tr>
<tr>
<td>$c_{89}$</td>
<td>$D_\theta CP$</td>
</tr>
<tr>
<td>$c_{49}$</td>
<td>$(2f_{12}/V)-I_{W_1}V\delta_o/(a_r O)$</td>
</tr>
<tr>
<td>$c_{55}$</td>
<td>$2(f_{33} a^2 + f_{22})/V$</td>
</tr>
<tr>
<td></td>
<td>$+2D_z d^2$</td>
</tr>
<tr>
<td>$c_{57}$</td>
<td>$I_{W_1}V a_1/(r_o a) - 2f_{12}/V$</td>
</tr>
<tr>
<td>$c_{59}$</td>
<td>$2(a^2 f_{33}+f_{22})/V$</td>
</tr>
<tr>
<td>$c_{66}$</td>
<td>$2D_{AX}+f_{33} r_o 2/V$</td>
</tr>
<tr>
<td>$c_{69}$</td>
<td>$2a_r O f_{33}/V$</td>
</tr>
<tr>
<td>$c_{74}$</td>
<td>$-2D_x P$</td>
</tr>
<tr>
<td>$c_{72}$</td>
<td>$-2D_x [(L/2)+b_o]$</td>
</tr>
<tr>
<td>$c_{81}$</td>
<td>$-2\alpha D_x P$</td>
</tr>
<tr>
<td>$c_{84}$</td>
<td>$2\alpha D_x P$</td>
</tr>
<tr>
<td>$c_{88}$</td>
<td>$D_\theta CP$</td>
</tr>
<tr>
<td>$c_{82}$</td>
<td>$-D_\theta CP$</td>
</tr>
<tr>
<td>Table C.1 (Cont’d)</td>
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<tr>
<td>-------------------</td>
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</tr>
<tr>
<td>( c_{8, 21} = D_{\theta , CP} )</td>
<td>( c_{9, 21} = D_{\theta , CP} )</td>
</tr>
<tr>
<td>( c_{9, 21} = D_{\theta , CP} + D_{\theta , W} )</td>
<td>( c_{9, 21} = D_{\theta , CP} )</td>
</tr>
<tr>
<td>( c_{10, 11} = (2f_{12}/V) - I_{W_1} V\delta_0/(r_0a) )</td>
<td>( c_{10, 16} = 2f_{11}/V )</td>
</tr>
<tr>
<td>( c_{10, 17} = 2f_{11}/V )</td>
<td>( c_{10, 18} = (2f_{12}/V) - I_{W_1} V\delta_0/(r_0a) )</td>
</tr>
<tr>
<td>( c_{11, 10} = I_{W_1} V\alpha_1/(r_0a) - 2f_{12}/V )</td>
<td>( c_{11, 11} = 2(f_{33}a^2+f_{22})/V + 2D_{Z_p} d^2 )</td>
</tr>
<tr>
<td>( c_{11, 12} = a r_0 f_{33}/V )</td>
<td>( c_{11, 16} = I_{W_1} V\alpha_1/(r_0a) - 2f_{12}/V )</td>
</tr>
<tr>
<td>( c_{11, 17} = \alpha[I_{W_1} V\alpha_1/(r_0a) - 2f_{12}/V] + 2(a^2f_{33} + f_{22})/V )</td>
<td>( c_{11, 18} = 2(a^2f_{33} + f_{22})/V )</td>
</tr>
<tr>
<td>( c_{12, 11} = 2a r_0 f_{33}/V )</td>
<td>( c_{12, 12} = 2D_{AX} + f_{33} r_0^{-2}/V )</td>
</tr>
<tr>
<td>( c_{12, 17} = 2a r_0 f_{33}/V )</td>
<td>( c_{12, 18} = 2a r_0 f_{33}/V )</td>
</tr>
<tr>
<td>( c_{13, 13} = 2D_{X_p} + 2f_{11}/V )</td>
<td>( c_{13, 14} = (2f_{12}/V) - I_{W_1} V\delta_0/(r_0a) )</td>
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<td>Table C.1 (Cont'd)</td>
<td></td>
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<tr>
<td>---------------------</td>
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<tr>
<td>( c_{13\ 16} = 2f_{11}/V )</td>
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</tr>
<tr>
<td>( c_{13\ 18} = (2f_{12}/V) - I_{w1} V\delta_o/(r_o a) )</td>
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</tr>
<tr>
<td>( c_{14\ 14} = 2(f_{33}a^2 + f_{22})/V + 2D_z d^2 )</td>
<td></td>
</tr>
<tr>
<td>( c_{14\ 16} = I_{w1} Va_1/(r_o a) - 2f_{12}/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{14\ 18} = 2(a^2f_{33} + f_{22})/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{15\ 15} = 2D_X + f_{33}r_o^2/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{15\ 18} = 2a r_0 f_{33}/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{16\ 13} = -2D_{xp} )</td>
<td></td>
</tr>
<tr>
<td>( c_{16\ 19} = -2D_X )</td>
<td></td>
</tr>
<tr>
<td>( c_{16\ 23} = -2hD_X )</td>
<td></td>
</tr>
<tr>
<td>( c_{17\ 11} = -2D_z d^2 )</td>
<td></td>
</tr>
<tr>
<td>( c_{17\ 14} = -2D_z d^2 )</td>
<td></td>
</tr>
<tr>
<td>( c_{13\ 17} = -2\epsilon f_{11}/V + 2f_{12}/V - I_{w1} V\delta_o/(r_o a) )</td>
<td></td>
</tr>
<tr>
<td>( c_{14\ 13} = I_{w1} Va_1/(r_o a) - 2f_{12}/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{14\ 15} = ar_0 f_{33}/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{14\ 17} = 2(f_{33}a^2 + f_{22})/V - \epsilon[I_{w1} Va_1/(r_o a) - 2f_{12}/V] )</td>
<td></td>
</tr>
<tr>
<td>( c_{15\ 14} = 2ar_0 f_{33}/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{15\ 17} = 2ar_0 f_{33}/V )</td>
<td></td>
</tr>
<tr>
<td>( c_{16\ 10} = -2D_{xp} )</td>
<td></td>
</tr>
<tr>
<td>( c_{16\ 16} = 2D_X )</td>
<td></td>
</tr>
<tr>
<td>( c_{16\ 20} = 2D_X [(L/2) + b_o] )</td>
<td></td>
</tr>
<tr>
<td>( c_{17\ 10} = -2\epsilon D_{xp} )</td>
<td></td>
</tr>
<tr>
<td>( c_{17\ 13} = 2\epsilon D_{xp} )</td>
<td></td>
</tr>
<tr>
<td>( c_{17\ 17} = D_\theta_{CP} )</td>
<td></td>
</tr>
</tbody>
</table>
TABLE C.1 (Cont'd)

\[
k_{19} = -2f_{11} - 2k_{xL}\ell
\]

\[
k_{22} = 2(f_{12} + k_{z}d^2) + k_{\theta} + k_{xL}\ell^2
\]

\[
k_{25} = -k_{\theta} + k_{xL}\ell^2
\]

\[
k_{28} = 2f_{12} + \ell[(2af_{33}\lambda_1/r_0) - 2f_{22}\Delta_1/(ar_0)]
\]

\[
k_{31} = 2f_{33}\lambda_1
\]

\[
k_{37} = 2f_{33}\lambda_1
\]

\[
k_{41} = -k_{xL}
\]

\[
k_{44} = [W_{App}(\Delta_1 + a_1)/a] - [2f_{12}\Delta_1/(ar_0)] + 2k_{xP} + k_{xL}
\]

\[
k_{47} = [W_{App}(\Delta_1 + a_1)/a] - [2f_{12}\Delta_1/r_0 a]
\]

\[
k_{49} = -2f_{11} + 2k_{xL}\ell
\]

\[
k_{54} = [(2f_{33}\lambda_1)/r_0] - [(2f_{22}\Delta_1)/(r_0 a)] + k_{xL}\ell
\]

\[
k_{21} = (2af_{33}\lambda_1/r_0)
\]

\[
-2f_{22}\Delta_1/(ar_0) - k_{xL}\ell
\]

\[
k_{24} = k_{xL}\ell
\]

\[
k_{27} = (2af_{33}\lambda_1/r_0)
\]

\[
-2f_{22}\Delta_1/(ar_0)
\]

\[
k_{29} = 2f_{12} + 2k_{xL}\ell^2
\]

\[
k_{33} = 2k_{AX}
\]

\[
k_{38} = 2xf_{33}\lambda_1
\]

\[
k_{42} = k_{xL}\ell
\]

\[
k_{45} = -2f_{11} + k_{xL}\ell
\]

\[
k_{48} = -[W_{App}(a_1 + \Delta_1)/a] - 2f_{12}\Delta_1/(r_0 a)]\ell - 2f_{11}
\]

\[
k_{55} = 2(f_{12} + k_{z}d^2) + k_{\theta} + k_{xL}\ell^2
\]
| $k_{57}$ | $= \left[ 2f_{33} \alpha_{11}/r_{0} \right] - \left[ 2f_{22} \Delta_{1}/(r_{0} a) \right]$ |
| $k_{58}$ | $= 2f_{12} - \lambda \left[ 2f_{33} \alpha_{11}/r_{0} \right] - \left[ 2f_{22} \Delta_{1}/(r_{0} a) \right]$ |
| $k_{59}$ | $= 2f_{12} + 2k_{x_{L}} x^{2}$ |
| $k_{60}$ | $= 2f_{33} \alpha_{11}$ |
| $k_{61}$ | $= 2f_{33} \alpha_{11}$ |
| $k_{62}$ | $= - \lambda (2f_{33} \alpha_{11})$ |
| $k_{63}$ | $= -2k_{x_{p}}$ |
| $k_{64}$ | $= 2k_{x}$ |
| $k_{65}$ | $= 2k_{x}$ |
| $k_{66}$ | $= 2k_{x_{p}}$ |
| $k_{67}$ | $= 2L/2 + b_{0}/2 k_{x}$ |
| $k_{68}$ | $= -2L/2 + b_{0}/2 k_{x}$ |
| $k_{69}$ | $= -2k_{z_{p}} d^{2}$ |
| $k_{70}$ | $= 2k_{z_{p}} d^{2}$ |
| $k_{71}$ | $= k_{\theta_{CP}}$ |
| $k_{72}$ | $= -k_{\theta_{CP}}$ |
| $k_{73}$ | $= k_{\theta_{CP}}$ |
| $k_{74}$ | $= -2k_{z_{p}} d^{2}$ |
| $k_{75}$ | $= 2k_{z_{p}} d^{2}$ |
| $k_{76}$ | $= k_{\theta_{CP}}$ |
| $k_{77}$ | $= -k_{\theta_{CP}}$ |
| $k_{78}$ | $= -k_{\theta_{CP}}$ |
| $k_{79}$ | $= 2k_{z_{p}} d^{2}$ |
| $k_{80}$ | $= k_{\theta_{CP}}$ |
| $k_{81}$ | $= k_{\theta_{CP}}$ |
| $k_{82}$ | $= k_{\theta_{CP}}$ |
| $k_{83}$ | $= k_{\theta_{CP}}$ |
| $k_{84}$ | $= k_{\theta_{CP}}$ |
| $k_{85}$ | $= k_{\theta_{CP}}$ |
| $k_{86}$ | $= k_{\theta_{CP}}$ |
| $k_{87}$ | $= k_{\theta_{CP}}$ |
| $k_{88}$ | $= k_{\theta_{CP}}$ |
| $k_{89}$ | $= k_{\theta_{CP}} + k_{\theta_{W}}$ |
| $k_{90}$ | $= -k_{\theta_{CP}}$ |
| $k_{91}$ | $= -k_{\theta_{CP}}$ |
| $k_{92}$ | $= -2k_{z_{p}} d^{2}$ |
| $k_{93}$ | $= 2k_{z_{p}} d^{2}$ |
| $k_{94}$ | $= k_{\theta_{CP}}$ |
| $k_{95}$ | $= k_{\theta_{CP}}$ |
| $k_{96}$ | $= k_{\theta_{CP}}$ |
| $k_{97}$ | $= k_{\theta_{CP}}$ |
| $k_{98}$ | $= k_{\theta_{CP}}$ |
| $k_{99}$ | $= k_{\theta_{CP}}$ |
| $k_{100}$ | $= -k_{\theta_{CP}}$ |
TABLE C.1 (Cont'd)

\[ k_{10\ 11} = -2f_{11} - k_{x_L} \]

\[ k_{10\ 14} = -k_{x_L} \]

\[ k_{10\ 17} = -2f_{11} + \varepsilon \left( \frac{W_{App}(\Delta_1 + a_1)}{a} \right) \]
\[ - \left[ \frac{2f_{12} \Delta_1}{(a r_0)} \right] \]

\[ k_{11\ 10} = \frac{2a f_3 \lambda_1}{r_0} - 2f_{22} \Delta_1 / (a r_0) \]
\[ - k_{x_L} \]

\[ k_{11\ 13} = k_{x_L} \]

\[ k_{11\ 16} = \frac{2a f_3 \lambda_1}{r_0} - 2f_{22} \Delta_1 / (a r_0) \]

\[ k_{11\ 18} = 2f_{12} + 2k_{x_L} \]

\[ k_{12\ 12} = 2k_{AX} \]

\[ k_{12\ 17} = 2k f_3 \lambda_1 \]

\[ k_{10\ 10} = \frac{W_{App}(\Delta_1 + a_1)}{a} \]
\[ - \left[ \frac{2f_{12} \Delta_1}{(a r_0)} \right] + 2k_{x_L} \]

\[ + k_{x_L} \]

\[ k_{10\ 13} = -k_{x_L} \]

\[ k_{10\ 16} = \frac{W_{App}(\Delta_1 + a_1)}{a} \]
\[ - \left( \frac{2f_{12} \Delta_1}{(a r_0)} \right) \]

\[ k_{10\ 18} = -2f_{11} - 2k_{x_L} \]

\[ k_{11\ 11} = 2f_{12} + k_{x_L} \]
\[ + k_{x_L} \]

\[ k_{11\ 14} = \frac{k_{x_L}}{a} \]

\[ k_{11\ 17} = 2f_{12} + \varepsilon \left( \frac{2a f_3 \lambda_1}{r_0} \right) \]
\[ - \left[ \frac{2f_{22} \Delta_1}{(a r_0)} \right] \]

\[ k_{11\ 19} = 2f_{33} \lambda_1 \]

\[ k_{12\ 10} = 2f_{33} \lambda_1 \]

\[ k_{12\ 16} = 2f_{33} \lambda_1 \]

\[ k_{13\ 13} = \frac{W_{App}(\Delta_1 + a_1)}{a} \]
\[ - \left[ \frac{2f_{12} \Delta_1}{(a r_0)} \right] + 2k_{x_L} \]

\[ + k_{x_L} \]
TABLE C.1 (Cont'd)

\[
\begin{align*}
k_{13\ 14} &= 2f_{11} + k_{xL} \frac{L}{2} \\
k_{13\ 17} &= -2f_{11} - \xi [W_{\text{APP}}(\Delta_1 + a_1)/a \\
&\quad -2f_{12}\Delta_1/(r_o a)] \\
k_{14\ 10} &= -k_{xL} \frac{L}{2} \\
k_{14\ 13} &= [2f_{33}\alpha_1 l/r_o] - [2f_{22}\Delta_1 / (r_o a)] \\
&\quad + k_{xL} \frac{L}{2} \\
k_{14\ 16} &= [2f_{33}\alpha_1 l/r_o] - [2f_{22}\Delta_1 / (r_o a)] \\
k_{14\ 18} &= 2f_{12} + 2k_{xL} \frac{L}{2} \\
k_{15\ 15} &= 2k_{AX} \\
k_{15\ 17} &= -\xi (2f_{33}\lambda_1) \\
k_{16\ 13} &= -2k_{xp} \\
k_{16\ 19} &= -2k_x \\
k_{16\ 16} &= 2k_x \\
k_{16\ 20} &= [(L/2) + b_0](2k_x)
\end{align*}
\]
$$\begin{align*}
\text{k}_{16} 23 &= -2hk_x \\
\text{k}_{17} 11 &= -2k_x d^2 \\
\text{k}_{17} 14 &= -2k_x d^2 \\
\text{k}_{17} 18 &= k_{\theta_{\text{CP}}} \\
\text{k}_{17} 21 &= -k_{\theta_{\text{CP}}} \\
\text{k}_{18} 14 &= -2k_x d^2 \\
\text{k}_{18} 18 &= k_{\theta_{\text{CP}}} + k_{\theta_{W}} \\
\text{k}_{18} 21 &= -k_{\theta_{\text{CP}}} \\
\text{k}_{19} 16 &= -2k_x \\
\text{k}_{19} 22 &= 2k_x(h+h_{\text{sp}}) \\
\text{k}_{20} 7 &= -2[(L/2) + b_o]k_x \\
\text{k}_{20} 9 &= -k_{\theta_{\text{CP}}} \\
\text{k}_{20} 17 &= -k_{\theta_{\text{CP}}} \\
\text{k}_{17} 10 &= -2k_x p \\
\text{k}_{17} 13 &= 2k_x p \\
\text{k}_{17} 17 &= k_{\theta_{\text{CP}}} \\
\text{k}_{17} 20 &= -k_{\theta_{\text{CP}}} \\
\text{k}_{18} 11 &= -2k_x d^2 \\
\text{k}_{18} 17 &= k_{\theta_{\text{CP}}} \\
\text{k}_{18} 20 &= -k_{\theta_{\text{CP}}} \\
\text{k}_{19} 7 &= -2k_x \\
\text{k}_{19} 19 &= 4k_x \\
\text{k}_{19} 23 &= 2k_x(h+h_{\text{sp}}) \\
\text{k}_{20} 8 &= -k_{\theta_{\text{CP}}} \\
\text{k}_{20} 16 &= 2[(L/2) + b_o]k_x \\
\text{k}_{20} 18 &= -k_{\theta_{\text{CP}}}
\end{align*}$$
TABLE C.1 (Cont'd)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{20\ 20} = (L+2b_0)^2k_x+2k_{\theta CP}$</td>
<td></td>
</tr>
<tr>
<td>$k_{20\ 23} = -2k_x<a href="h+h_%7Bsp%7D">(L/2) + b_0</a>$</td>
<td></td>
</tr>
<tr>
<td>$k_{21\ 8} = k_{\theta CP}$</td>
<td></td>
</tr>
<tr>
<td>$k_{21\ 16} = 2b_0k_x$</td>
<td></td>
</tr>
<tr>
<td>$k_{21\ 18} = -k_{\theta CP}$</td>
<td></td>
</tr>
<tr>
<td>$k_{21\ 21} = 2k_{\theta CP} + 4k_{FLEX}$</td>
<td></td>
</tr>
<tr>
<td>$k_{21\ 23} = -2b_0k_x(h+h_{sp})$</td>
<td></td>
</tr>
<tr>
<td>$k_{22\ 4} = -k_yd^2a_1/a - k_{sp}a_1/a$</td>
<td></td>
</tr>
<tr>
<td>$k_{22\ 19} = 2k_x(h_{sp} + h)$</td>
<td></td>
</tr>
<tr>
<td>$k_{22\ 22} = 2k_x(h_{sp} + h)^2 + 2k_yd^2 + k_{TOR} + 2k_{sp} - mcgh_{CG}/2$</td>
<td></td>
</tr>
<tr>
<td>$k_{23\ 10} = -k_yd^2a_1/a - k_{sp}a_1/a$</td>
<td></td>
</tr>
<tr>
<td>$k_{23\ 16} = -2k_x(h+h_{sp}) - 2k_yd^2a_1/a$</td>
<td></td>
</tr>
<tr>
<td>$k_{23\ 19} = 2k_x(h_{sp} + h)$</td>
<td></td>
</tr>
<tr>
<td>$k_{23\ 20} = 2k_x[(h+h_{sp})(L/2) + hb_0]$</td>
<td></td>
</tr>
<tr>
<td>$k_{23\ 22} = -2b_0k_x(h+h_{sp})$</td>
<td></td>
</tr>
<tr>
<td>$k_{23\ 23} = -k_{TOR}$</td>
<td></td>
</tr>
<tr>
<td>$k_{24\ 1} = -k_yd^2a_1/a - k_{sp}a_1/a$</td>
<td></td>
</tr>
<tr>
<td>$k_{24\ 7} = -2k_x(h+h_{sp}) - 2k_yd^2a_1/a - (2k_{sp}a_1/a)$</td>
<td></td>
</tr>
</tbody>
</table>
TABLE C.1 (Cont'd)

\[
\begin{align*}
    k_{23\ 20} &= -2k_x[(h+h_{sp})(L/2)+h_b] \quad & k_{23\ 22} &= -k_{TOR} \\
    k_{23\ 23} &= 2k_x(h_{sp}+h)^2 + 2k_yd^2 - m_cgh_{CG}/2 + k_{TOR} + 2k_{sp}
\end{align*}
\]
LIST OF SYMBOLS

\begin{itemize}
  \item \( a \): One half track gauge, ft.
  \item \( \tilde{a}_{c.m.} \): Absolute acceleration of center of mass of each half-car body
  \item \( \tilde{a}_o \): Absolute acceleration of point \( o \) on each half-car body
  \item \( b_o \): Longitudinal distance from half car body centroid to truck centerplate, ft.
  \item \( D_{AX} \): Axle torsional damping coefficient, \( \frac{\text{ft-lb-sec}}{\text{rad}} \)
  \item \( D_{FLEX} \): Lateral damping coefficient for car bending, \( \frac{\text{lb sec}}{\text{ft}} \)
  \item \( D_{TOR} \): Torsional damping coefficient for car body bending, \( \frac{\text{ft-lb-sec}}{\text{rad}} \)
  \item \( D_{sp} \): Roll damping coefficient due to fixed-ends of secondary spring groups, \( \frac{\text{ft-lb-sec}}{\text{rad}} \)
  \item \( D_x \): Secondary suspension lateral damping coefficient, \( \frac{\text{lb sec}}{\text{ft}} \)
  \item \( D_{xp} \): Primary suspension lateral damping coefficient, \( \frac{\text{lb sec}}{\text{ft}} \)
  \item \( D_y \): Secondary suspension vertical damping coefficient, \( \frac{\text{lb sec}}{\text{ft}} \)
  \item \( D_{zp} \): Primary suspension longitudinal damping coefficient, \( \frac{\text{lb sec}}{\text{ft}} \)
  \item \( D_{\theta CP} \): Yaw damping coefficient for centerplate connection, \( \frac{\text{ft-lb-sec}}{\text{rad}} \)
  \item \( D_{\theta L} \): Yaw damping coefficient for wheelset interconnection, \( \frac{\text{ft-lb-sec}}{\text{rad}} \)
  \item \( D_{\theta p} \): Primary suspension yaw damping coefficient, \( \frac{\text{ft-lb-sec}}{\text{rad}} \)
  \item \( D_{\theta w} \): Secondary suspension warp damping coefficient, \( \frac{\text{ft-lb-sec}}{\text{rad}} \)
  \item \( d \): One half distance between sideframes, ft
  \item \( d_0 \): Axle diameter, ft
  \item \( \hat{e}_{1L}, \hat{e}_{2L}, \hat{e}_{3L} \): Unit vectors at left wheel contact point (Figure A.4)
\end{itemize}
\( \hat{e}_{1R}, \hat{e}_{2R}, \hat{e}_{3R} \)
Unit vectors at right wheel contact point (Figure A.4)

\( \overline{F}_L, \overline{F}_R \)
Creep force on left and right wheels, respectively, lb

\( F_{L_x}, F_{L_y}, F_{L_z} \)
Components of \( \overline{F}_L \) along \( \hat{i}^{'}, \hat{j}^{'}, \hat{k}^{'1} \) (Figure A.2)

\( F_{R_x}, F_{R_y}, F_{R_z} \)
Components of \( \overline{F}_R \) along \( \hat{i}^{'}, \hat{j}^{'}, \hat{k}^{'1} \) (Figure A.2)

\( \overline{F}_s \)
Total suspension force acting on wheelset, lb

\( F_{s_x} \)
Component of \( \overline{F}_s \) along \( \hat{i}^{'1} \) (Figure A.2)

\( f_c \)
Frequency of least-damped mode at critical speed, Hz

\( f_{11}, f_{33} \)
Lateral and longitudinal creep coefficients, lb/wheel

\( f_{12}, f_{22} \)
Lateral/spin and spin creep coefficients, ft-lb/wheel and ft\(^2\)-lb/wheel, respectively

\( G \)
Shear modulus for steel, \( 1.656 \times 10^9 \frac{\text{lb}}{\text{ft}^2} \)

\( g \)
Acceleration due to gravity, \( 32.2 \frac{\text{ft}}{\text{sec}^2} \)

\( H_B \)
Angular momentum of bolster about its centroid, \( \frac{\text{slug ft}^2}{\text{sec}} \)

\( H_C \)
Angular momentum of car body about its centroid, \( \frac{\text{slug ft}^2}{\text{sec}} \)

\( H_{CF} \)
Angular momentum of front half-car body about its centroid, \( \frac{\text{slug ft}^2}{\text{sec}} \)

\( H_{CR} \)
Angular momentum of rear half-car body about its centroid, \( \frac{\text{slug ft}^2}{\text{sec}} \)

\( H_G \)
Angular momentum of wheelset about its centroid, \( \frac{\text{slug ft}^2}{\text{sec}} \)

\( H_{GL} \)
Angular momentum of left wheel about the wheelset centroid, \( \frac{\text{slug ft}^2}{\text{sec}} \)
\( \bar{H}_{GR} \) Angular momentum of right wheel about the wheelset centroid, \( \text{slug ft}^2 \text{sec}^{-1} \)

\( \bar{H}_s \) Angular momentum of sideframe about its centroid, \( \text{slug ft}^2 \text{sec}^{-1} \)

\( \bar{H}_o \) Angular momentum of each half-car about point \( o \)

\( h \) Height of "hinge point" of car body above bolster, ft

\( h_{CP} \) Height of car body centroid above bolster, ft

\( h_{SP} \) Secondary suspension spring height at equilibrium, ft

\( \bar{T}_B \) Mass moment of inertia of bolster about its centroid, \( \text{slug ft}^2 \)

\( I_{B_1}, I_{B_2} \) Components of \( \bar{T}_B \) about \( (\hat{i}''', \hat{j}''', \hat{k}''') \)

\( I_{B_3} \)

\( \bar{T}_C \) Mass moment of inertia of pseudo car body (Appendix B) and of complete car body (Appendix C) about its centroid, \( \text{slug ft}^2 \)

\( I_{C_1}, I_{C_2} \) Components of \( \bar{T}_C \) about \( (\hat{i}''', \hat{j}''', \hat{k}''') \)

\( I_{C_3} \)

\( \bar{T}_{CF} \) Mass moment of inertia of front half-car body about its centroid, \( \text{slug ft}^2 \)

\( \bar{T}_{CR} \) Mass moment of inertia of rear half-car body about its centroid, \( \text{slug ft}^2 \)

\( \bar{T}_s \) Mass moment of inertia of sideframe about its centroid, \( \text{slug ft}^2 \)

\( I_{S_1}, I_{S_2} \) Components of \( \bar{T}_s \) about \( (\hat{i}''', \hat{j}''', \hat{k}''') \)

\( I_{S_3} \)

\( \bar{T}_w \) Mass moment of inertia of wheelset about its centroid, \( \text{slug ft}^2 \)

\( I_{W_1}, I_{W_2} \) Components of \( \bar{T}_w \) about \( (\hat{i}''', \hat{j}''') \)
\( I_x, I_y, I_z \) Components of inertia tensor with respect to body axes of half-car bodies through point o

\( J_{yx}, J_{yz}, J_{xz} \)

\( \hat{i}', \hat{j}', \hat{k}' \) Unit vectors for wheelset body axes (Figure A.1)

\( \hat{i}'', \hat{j}'', \hat{k}'' \) Unit vectors for wheelset intermediate axes (Figure A.1)

\( \hat{k}''' \)

\( \hat{i'''}, \hat{j''', \hat{k}''' \) Unit vectors for equilibrium axis system (Figure A.1)

\( J_{yx c.m.}, J_{yz c.m.}, J_{xz c.m.} \)

\( k_{AX} \) Axle torsional stiffness coefficient, \( \text{ft lb rad}^{-1} \)

\( k_{FLEX} \) Torsional stiffness coefficient for car body lateral bending, \( \text{ft lb rad}^{-1} \)

\( k_{TOR} \) Torsional stiffness coefficient for car body torsional bending, \( \text{ft lb rad}^{-1} \)

\( k_{SP} \) Roll spring constant due to fixed ends of secondary spring groups, \( \text{ft lb rad}^{-1} \)

\( k_x \) Secondary suspension lateral spring constant, \( \frac{\text{lb}}{\text{ft}} \)

\( k_{x_L} \) Lateral spring constant for wheelset interconnection, \( \frac{\text{lb}}{\text{ft}} \)

\( k_{x_P} \) Primary suspension lateral spring constant, \( \frac{\text{lb}}{\text{ft}} \)

\( k_y \) Secondary suspension vertical spring constant, \( \frac{\text{lb}}{\text{ft}} \)
Primary suspension longitudinal spring constant, \( \frac{1b}{ft} \)

Yaw spring constant for centerplate connection, \( \frac{ft \: lb}{rad} \)

Yaw spring constant for wheelset interconnection, \( \frac{ft \: lb}{rad} \)

Secondary suspension warp spring constant, \( \frac{ft \: lb}{rad} \)

Longitudinal distance between hinge point and half-car body centroid, ft

Axle length, ft

Axle torsional moment, ft lb (Figure A.3)

Vector creep moment on left and right wheels, respectively ft lb

Components of \( \overline{M}_L \) acting about \( \hat{i}'' \) and \( \hat{j}'' \) directions (Figure A.2).

Components of \( \overline{M}_R \) acting about \( \hat{i}''' \) and \( \hat{j}''' \) directions (Figure A.2).

Bolster mass, slugs

Car body mass, slugs

Mass of one sideframe, slugs

Wheelset mass, slugs

Position vector of pseudo car body centroid

Position vector of wheelset centroid.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{r}<em>{CP_F}, \vec{r}</em>{CP_R} )</td>
<td>Position vectors of front and rear bolster centroids, respectively</td>
</tr>
<tr>
<td>( r_L, r_R )</td>
<td>Instantaneous rolling radii of left and right wheels, respectively</td>
</tr>
<tr>
<td>( \vec{r}<em>{LC}, \vec{r}</em>{RC} )</td>
<td>Position vector of left wheel contact point</td>
</tr>
<tr>
<td>( \vec{r}<em>{LS}, \vec{r}</em>{RS} )</td>
<td>Position vector of right wheel contact point</td>
</tr>
<tr>
<td>( \vec{r}<em>{LS}, \vec{r}</em>{RS} )</td>
<td>Position vectors of left and right sideframes, respectively</td>
</tr>
<tr>
<td>( r_o )</td>
<td>Wheel rolling radius at equilibrium, ft</td>
</tr>
<tr>
<td>( t )</td>
<td>Time, sec</td>
</tr>
<tr>
<td>( V )</td>
<td>Forward speed at equilibrium, fps</td>
</tr>
<tr>
<td>( V_c )</td>
<td>Critical speed, fps</td>
</tr>
<tr>
<td>( W_{\text{APP}} )</td>
<td>Axle load, lb (( W_{\text{APP}} = W'_{\text{APP}} + m_w g ))</td>
</tr>
<tr>
<td>( W'_{\text{APP}} )</td>
<td>Applied load to wheelset, lb (Figure A.2)</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>Lateral, vertical and longitudinal displacements of wheelset centroid for three degree-of-freedom wheelset model, ft</td>
</tr>
<tr>
<td>( x_B, y_B, z_B )</td>
<td>Lateral, vertical and longitudinal displacements of bolster centroid, ft</td>
</tr>
<tr>
<td>( x_c, y_c, z_c )</td>
<td>Lateral, vertical and longitudinal displacements of pseudo car body centroid, ft</td>
</tr>
<tr>
<td>( x_{CF}, x_{CR} )</td>
<td>Lateral displacements of front and rear half-car bodies, respectively, at the height of the hinge, ft</td>
</tr>
<tr>
<td>( x_{\text{FLEX}} )</td>
<td>Lateral displacement of hinge point, ft</td>
</tr>
<tr>
<td>( x_{\text{RIG}} )</td>
<td>Lateral displacement of imaginary line connecting truck centers at the height of the hinge, ft</td>
</tr>
<tr>
<td>( x_T )</td>
<td>Lateral displacement of truck centroid, ft</td>
</tr>
<tr>
<td>( x_{W1}, x_{W2} )</td>
<td>Lateral displacement of front and rear wheelset centroids on front truck relative to truck centerline, ft</td>
</tr>
<tr>
<td>( x_{W3}, x_{W4} )</td>
<td>Lateral displacements of front and rear wheelset centroids rear truck relative to truck centerline, ft</td>
</tr>
<tr>
<td>( y_{LS}, z_{LS} )</td>
<td>Vertical and longitudinal displacements of left sideframe, ft</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\gamma_{rs}$, $z_{rs}$</td>
<td>Vertical and longitudinal displacements of right side-frame, ft</td>
</tr>
<tr>
<td>$z_0$</td>
<td>Distance wheelset traveled along track from fixed point to equilibrium position, ft</td>
</tr>
<tr>
<td>$\alpha_j$</td>
<td>Real part of $j$-th eigenvalue</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Net torsional displacement of wheelset model about $i'$ axis, rad</td>
</tr>
<tr>
<td>$\beta_L$, $\beta_R$</td>
<td>Torsional displacements of left and right wheels, respectively about $i'$ axis, rad</td>
</tr>
<tr>
<td>$\beta_1$, $\beta_2$</td>
<td>Net torsional displacements of front and rear wheelsets on front truck about $i'$ axis, rad</td>
</tr>
<tr>
<td>$\beta_3$, $\beta_4$</td>
<td>Net torsional displacements of front and rear wheelsets on rear truck about $i'$ axis, rad</td>
</tr>
<tr>
<td>$\gamma_j$</td>
<td>$j$-th coefficient of characteristic polynomial</td>
</tr>
<tr>
<td>$\Delta_L$, $\Delta_R$</td>
<td>Lateral displacement of contact points on the left and right rails, respectively, ft</td>
</tr>
<tr>
<td>$\Delta_0$, $\Delta_1$</td>
<td>Coefficients of first two terms in series expansion of contact angle difference function</td>
</tr>
<tr>
<td>$\delta_0$, $\delta_1$</td>
<td>Coefficients of first two terms in series expansion of average contact angle function</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Damping ratio</td>
</tr>
<tr>
<td>$\eta_j$</td>
<td>Imaginary part of $j$-th eigenvalue</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Yaw displacement of wheelset model about $j'''$ axis, rad</td>
</tr>
<tr>
<td>$\theta_{CF}$, $\theta_{CF}$</td>
<td>Yaw displacements of front and rear half-car bodies about $j'''$ axis, rad</td>
</tr>
<tr>
<td>$\theta_{FLEX}$</td>
<td>Yaw displacement of half-car bodies about $j'''$ axis and relative to imaginary line between truck centers, rad</td>
</tr>
</tbody>
</table>
\( \theta_{\text{RIG}} \) Yaw displacement of imaginary line between truck centers about \( j'''' \) axis rad

\( \theta_T \) Yaw displacement of truck frame about \( j'''' \) axis, rad

\( \theta_W \) Warp displacement of truck frame about \( j'''' \) axis, rad

\( \theta_{w1}, \theta_{w2} \) Yaw displacements of front and rear wheelsets on front truck about \( j'''' \) and relative to line joining ends of sideframes when truck is distorted, rad

\( \theta_{w3}, \theta_{w4} \) Yaw displacements of front and rear wheelsets on rear truck about \( j'''' \) and relative to line joining ends of sideframes when truck is distorted, rad

\( \lambda_1 \) Effective wheel conicity, rad

\( \mu \) Coefficient of coulomb friction

\( \phi \) Roll displacement of wheelset model about \( k'''' \) axis, rad

\( \phi_C \) Roll displacement of pseudo car body about \( k'''' \) axis, rad

\( \phi_{cF}, \phi_{cR} \) Roll displacements of front and rear half-car bodies about \( k'''' \) axis, rad

\( \Omega \) Equilibrium angular velocity of wheelset model, rad/sec

\( \bar{\omega} \) Angular velocity of wheelset, rad/sec

\( \bar{\omega}_{\text{axes}} \) Angular velocity of an axis system, rad/sec

\( \bar{\omega}_B \) Angular velocity of bolster, rad/sec

\( \bar{\omega}_{cF}, \bar{\omega}_{cR} \) Angular velocities of front and rear half-car bodies, respectively, rad/sec

\( \bar{\omega}_L, \bar{\omega}_R \) Angular velocities of left and right wheels, respectively, rad/sec

\( \bar{\omega}_L, \bar{\omega}_R \) Angular velocities of left and right wheel contact points, respectively, rad/sec

\( \bar{\omega}_S \) Angular velocity of sideframes, rad/sec

\( \omega_x, \omega_y, \omega_z \) Components of body angular velocity of each half-car body about the body axes
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