RAIL-WHEEL GEOMETRY ASSOCIATED
WITH CONTACT STRESS ANALYSIS

TECHNICAL REPORT NO. 6

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This report records the derivation of a number of results pertaining to wheel and rail geometry that are needed for the analysis of contact stresses and rolling-creepage phenomena. In particular, results utilized in the authors' computer programs COUNTACT (for COUNTERformal contACT problems) and CONFORM (for CONFORMal contact problems) are given.

It is shown how the profile curves specified by engineering drawings for standard wheels and rails may be analyzed to find appropriate parameters needed to express the pertinent equations in the various coordinate systems utilized in contact stress analysis. For arbitrarily selected points of initial contact on the wheel tread and on the railhead, it is shown how to determine the feasibility of such contact, and how to determine the mutual separation of points on the two surfaces. It is also shown how to determine the curve of interpenetration which is used as an initial estimate of the contact patch boundary associated with a given relative approach (due to elastic deformation) of the loaded wheel and rail. The basis of a computer program (MIDSEP) to determine this separation is described.
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1. INTRODUCTION

The purpose of this report is to provide the derivation for a number of results pertaining to wheel and rail geometry that are needed for the analysis of contact stresses and friction-creepage phenomena. In particular, we wish to record here results that are utilized in the computer programs COUNTACT (Paul and Hashemi, 1977-a,b) and CONFORM (Paul and Hashemi, 1978, 1979).

2. STATEMENT OF THE PROBLEM

The geometry of the wheels and rails for railroad vehicles are usually specified in drawings of so-called profile curves in the plane of symmetry of the wheel or the rail. In Sec. 3 and in Appendix 1 we show how these drawings can be used to provide parameters and equations required for stress analysis purposes.

Rails and wheels can have one or more contact points, referred to as initial contact points, prior to any deformation. The initial contact points depend on the orientation of the wheelset relative to the track. For sufficiently small yaw angles of the wheelset, the initial contact points are located in a midplane. Only the case of a single initial contact point is discussed in this paper.

For a given position of the wheel relative to the rail, it is required to determine the initial contact points. This is discussed in Sections 4 and 5.

For a feasible initial contact point, the distance between two points, in the direction of the normal to the rail or wheel at the initial contact point, on the rail and wheel, is required. This distance is called the initial separation, discussed in Sec. 6.

*The plane which passes through the wheel axis and the contact point will be referred to as the midplane.

**For a discussion of how the initial contact points (single or multiple) are influenced by lateral wheelset motions see Cooperider, et. al. (1975) and Heller and Cooperider (1977).
The region where the rail and wheel touch, after load is applied, is called contact patch. The determination of an initial estimate for the boundary of the contact patch is discussed in Sec. 7.

3. RAIL AND WHEEL PROFILES

The railhead profile is the curve intersected by the rail surface and a plane normal to the rail axis [see Fig.(1a)]. The wheel profile is the curve of intersection of the wheel surface and a plane through the wheel axis. Standard American wheels and rails are manufactured with profiles consisting exclusively of straight line segments and circular arc segments. To write algebraic expressions for rail and wheel profiles, the following parameters, associated with a coordinate system $(x,z)$ fixed in the wheel (or rail), are needed (see Figs. 2 and 3):

1. The slope $a_i$ and $z$-intercept $b_i$ of straight line segment $i$
2. The radius $R_i$ and arc-center coordinates $(a_i, b_i)$ of circular arc segment $i$
3. The "end point" coordinates $(x_i, y_i)$ of the right hand end of segment $i$

In any specific case, information must be provided to calculate the above mentioned parameters (see Appendix 1 for derivation of the parameters associated with the wheel and rail of Figs. 2 and 3).

In the segment $i$, defined by

$$x_{i-1} < x < x_i$$

(3.1)

the "profile equations" (for wheel or rail) are:

$$z = b_i + [R_i^2 - (x - a_i)^2]^{1/2} \quad (arc \ segment)$$

(3.2-a)

$$z = a_i x + b_i \quad (straight \ segment)$$

(3.2-b)

When necessary, a superscript $R$ or $W$ will be used to distinguish between the coordinates $(x^R, z^R)$ of the rail and the coordinates $(x^W, z^W)$ of the wheel.
Fig. 1 (a) Wheel profile  
(b) Rail profile
Fig. 2. Wheel profile curves
(a) Parameters referred to wheel reference axes $(x^W, z^W)$
(b) Parameters referred to global axes $(\xi, \zeta)$
For the usual case of symmetric rails, it is convenient to locate the 
z-axis parallel to the axis of symmetry of the cross-section (see Fig. 3). 
The x-axis of the rail is conveniently chosen to be transverse to the rail.

4. FEASIBILITY OF INITIAL CONTACT PATCH

The rail and wheel can have one or more initial contact points prior to 
deforming under the action of the applied loads—depending upon the orientation of the wheelset relative to the rails. For sufficiently small yaw 
angles of the wheelset, the initial contact points are located in a plane 
which passes through the wheel axis and is normal to the rail surface. 
This plane coincides with the midplane, defined in Sec. 2.

This discussion is limited to the case where there exists a single point 
of initial contact between wheel and rail.

The initial contact point C on the wheel is determined by its x-coordinate 
\( x^W_C \) (see Fig. 2). Similarly, the initial contact point on the rail is 
determined by \( x^R_C \) (see Fig. 3).

For given locations of C on the wheel and on the rail, let 
\((\xi, \eta, \zeta)\) be a coordinate system with origin at point C, with the \( \zeta \)-axis along 
the common normal, and with the \( \eta \)-axis normal to the midplane, as illustrated 
in Fig. 4. The initial contact point C is feasible if, and only if, the 
distance between any two points on the rail and wheel profiles, which have the 
same values of \( \xi \) and \( \eta \) is positive; i.e. when 

\[
    s = \zeta^W(\xi, \eta) - \zeta^R(\xi, \eta) \geq 0 
\]  (4.1)

We now need to write the transformation equations between the wheel and rail 
coordinates \((x,z)\) and the global coordinates \((\xi,\zeta)\) for an arbitrary point P. 
If \( \theta_C \) is the angle through which the \( \xi \)-axis is rotated (positive counterclockwise) 
with respect to the x-axis and \((x_C, z_C)\) are the coordinates of point C, in the 
x-z coordinate system, then it follows from Fig. 5 that:
Fig. 3. Rail profile
(a) in rail reference system \((x^R, z^R)\)
(b) in global reference system \((\xi, \zeta)\)
Fig. 4. (a) Cross section through wheel axis and contact point C  
(b) Projection of contact patch on plane $\zeta = 0$  
(c) Cross section through plane $\xi = 0$

Fig. 5. Transformation from $(x,z)$ to $(\xi,\zeta)$ axes
\[ \xi = (x-x_c) \cos \theta_c + (z-z_c) \sin \theta_c \] (4.2-a)

\[ \zeta = (z-z_c) \cos \theta_c - (x-x_c) \sin \theta_c \] (4.2-b)

or

\[ x = x_c + \xi \cos \theta_c - \zeta \sin \theta_c \] (4.3-a)

\[ z = z_c + \xi \sin \theta_c + \zeta \cos \theta_c \] (4.3-b)

Substituting Eqs. (4.3) into Eqs. (3.2) and solving for \( \zeta \), we obtain the transformed equations (4.5-a,b) of the profile, in the range

\[ \xi_{i-1} < \xi < \xi_i \] (4.4)

\[ \zeta = \begin{cases} 
\beta_i + \frac{R_i}{2} - (\xi - a_i)^2 \right)^{1/2} & \text{(arc segments)} \\
\alpha_i \xi + \beta_i & \text{(straight segments)}
\end{cases} \] (4.5-a)

where:

\[ \alpha_i = \begin{cases} 
(a_i-x_c) \cos \theta_c + (b_i-z_c) \sin \theta_c & \text{(arc)} \\
\frac{a_i \cos \theta_c - \sin \theta_c}{a_i \sin \theta_c + \cos \theta_c} & \text{(straight)}
\end{cases} \] (4.6-a)

Similarly:

\[ \beta_i = \begin{cases} 
(b_i-z_c) \cos \theta_c - (a_i-x_c) \sin \theta_c & \text{(arc)} \\
\frac{(a_i x_c + b_i-z_c)/(\cos \theta_c + a_i \sin \theta_c)} & \text{(straight)}
\end{cases} \] (4.7-a)

and

\[ \xi_i = (x_i-x_c) \cos \theta_c + (z_i-z_c) \sin \theta_c \] (4.8)

\( \xi_i \) is the segment end point abscissa in the \((\xi, \zeta)\) system.
For a given value of $x_c$, the appropriate segment number $i$ is found from the range restriction

$$x_{i-1} < x_c < x_i$$  \hspace{.5cm} (4.9)

and the corresponding value of $z_c$ is given by Eqs. (3.2) in the form

$$z_c = \begin{cases} 
  b_i + \left[ R_i^2 - (x_c - a_i)^2 \right]^{1/2} & \text{(arc)} \\
  a_i x_c + b_i & \text{(straight)} 
\end{cases}$$  \hspace{.5cm} (4.10)

From Figure 5, we see that the profile slope at point $C$ is given by

$$\left( \frac{dz}{dx} \right)_C = \tan \theta_c$$  \hspace{.5cm} (4.11)

Hence the angle $\theta_c$ is given by

$$\theta_c = \tan^{-1} \left[ \frac{- (x - a_i)}{\left[ R_i^2 - (x_0 - a_i)^2 \right]^{1/2}} \right] \hspace{.5cm} (arc)$$  \hspace{.5cm} (4.12-a)

$$\theta_c = \tan^{-1} \left( \frac{1}{a_i} \right) \hspace{.5cm} (\text{straight})$$  \hspace{.5cm} (4.12-b)

Therefore, with $(a_i, b_i), R_i, x_i$ and $x_c$ given for the rail (wheel), $\zeta^R (\zeta^W)$ is obtained, for a given value of $\xi$, by the following procedure:

(a) calculate $z_c$ and $\theta_c$ using Eqs. (4.10) and (4.12) respectively.

(b) From Eqs. (4.6), (4.7) and (4.8), calculate $(\alpha_i, \beta_i)$ and $\xi_i$ respectively.

(c) Finally, calculate $\zeta^R (\zeta^W)$ from Eqs. (4.5).
5. **OUTLINE OF COMPUTER PROGRAM MIDSEP**

Based on the above equations and procedures, a FORTRAN program* called MIDSEP (MIDplane SEParation) was written to provide the following as output:**

(a) Transformed rail and wheel parameters: $\alpha_i$, $\beta_i$, $\xi_i$ (see Figs. 2 and 3)
(b) Value of the separation function $\Delta \xi = \xi^W - \xi^R$ on the plane $n = 0$, at equally spaced values of $\xi$.

The program is organized as shown in Fig. 6. A brief description of each program block follows.

**Main Program (MIDSEP):**

The purpose of the main program is to manage input and output, to call appropriate subprograms as needed, and to interlink the various components needed for overall program logic. It calls upon subroutines RAILO and WHEELO to calculate $\xi_c$, $\theta_c$, $\alpha_i$, $\beta_i$, $\xi_i$, for rail and wheel. Then it will continue to calculate $\xi^R$ and $\xi^W$ for equal increments of $\xi$, by calling subroutines RAIL and WHEEL. Finally it calculates $\Delta \xi = \xi^W - \xi^R$.

Subroutine RAILO: Calculates the $z$-coordinate of the initial contact point on the rail, and the transformed parameters $(\alpha_i$, $\beta_i$, $\xi_i)$ needed for the rail.

Subroutine WHEELO: Calculates the $z$-coordinate of the initial contact point on wheel and the transformed parameters $(\alpha_i$, $\beta_i$, $\xi_i)$ needed for the wheel.

Subfunction RAIL: Calculates the $\xi^R$ for any $\xi$; i.e. the rail profile in the global coordinate system $(\xi, \xi)$ of the midplane.

Subfunction WHEEL: Calculates the $\xi^W$ for any $\xi$; i.e. the wheel profile in the global coordinate system $(\xi, \xi)$ of the midplane.

A numerical example which fully illustrates the use of Program MIDSEP is given in Appendix 2 of Paul and Hashemi [1978].

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**The output described here serves as input for the calculations of Section 6."
Fig. 6. Organization of program MIDSEP. Arrows point from calling to called program.
6. NUMERICAL DETERMINATION OF INITIAL SEPARATION FUNCTION

The initial separation function is the distance between two points in the \( \zeta \)-direction. It is obtained from the equation

\[
\Delta \zeta = \zeta^W - \zeta^R
\]

(6.1)

where \( \zeta^W \) is the wheel shape function and \( \zeta^R \) is the rail shape function, both corresponding to the same value of \( \zeta \).

6.1 Rail Shape Function and Unit Normal Vector

The rail is a cylindrical surface. Therefore, in the global coordinate system \((\xi, \eta, \zeta)\) its shape function will be the same as its profile function given by equations (4.5), i.e.

\[
\zeta^R = \begin{cases} 
\alpha_i \xi + \beta_i & \text{(straight)} \\
\beta_i + \left[ R_i^2 - (\xi - \alpha_i)^2 \right]^{1/2} & \text{(arc)}
\end{cases}
\]

(6.2)

where \( \alpha_i \), \( \beta_i \) and \( \xi_i \) are given by Eqs. (4.6-4.8). The components of the unit vector \( \mathbf{n} \), normal to the rail surface, are obtained as follows:

\[
\mathbf{n} = (n_\xi, n_\eta, n_\zeta) = \left( \frac{\partial F}{\partial \xi}, 0, \frac{\partial F}{\partial \zeta} \right) / N
\]

(6.3)

where

\[
N = \left[ \left( \frac{\partial F}{\partial \xi} \right)^2 + \left( \frac{\partial F}{\partial \zeta} \right)^2 \right]^{1/2}
\]

(6.4)

and

\[
F(\xi, \zeta) = \begin{cases} 
\beta_i + \left[ R_i^2 - (\xi - \alpha_i)^2 \right]^{1/2} - \zeta & \text{(arc)} \\
\alpha_i \xi + \beta_i - \zeta & \text{(straight)}
\end{cases}
\]

(6.5)

Therefore,
\[
\frac{dF}{d\xi} = \begin{cases} 
\frac{-(\xi - \alpha_i)}{[R_i^2 - (\xi - \alpha_i)^2]^{1/2}} & \text{(arc)} \\
\alpha_i & \text{(straight)} 
\end{cases} 
\]

\[
\frac{dF}{d\zeta} = -1 
\]

\[
N = \begin{cases} 
\frac{R_i}{[R_i^2 - (\xi - \alpha_i)^2]^{1/2}} & \text{(arc)} \\
[1 + \alpha_i^2]^{1/2} & \text{(straight)} 
\end{cases} 
\]

\[
n_{\xi} = \begin{cases} 
\frac{-(\xi - \alpha_i)}{R_i} & \text{(arc)} \\
\alpha_i[1 + \alpha_i^2]^{-1/2} & \text{(straight)} 
\end{cases} 
\]

\[
n_{\zeta} = \begin{cases} 
\frac{(\xi - \beta_i)}{R_i} & \text{(arc)} \\
[1 + \alpha_i^2]^{-1/2} & \text{(straight)} 
\end{cases} 
\]

6.2 Wheel Shape Function

The wheel is a surface of revolution with nominal radius \(R_W\) (the distance from the wheel axis to the origin 0). Its surface equation in the wheel coordinate system \((x_W, y_W, z_W)\) in the domain \(x_i-1 \leq x \leq x_i\) is (see Fig. 7):

\[
(R_W - z_W)^2 - (\rho^2 - y^2) = 0 
\]

where

\[
\rho = R_W - z_W = \begin{cases} 
R_W - (\sqrt{R_i^2 - (x-a_i)^2} + b_i) & \text{(arc)} \\
R_W - (a_i x + b_i) & \text{(straight)} 
\end{cases} 
\]

Therefore,
Fig. 7 Rail and wheel profile functions
We may express $z_W$ in terms of $\xi$ and $\eta$ if we note that $y = \eta$, and $x$ is given by Eq. (4.3-a).

From Fig. 7, it is apparent that

$$F(\xi, \eta, \zeta_W) = (z_W - R_W^2 - (\rho^2 - \eta^2) = 0 \quad (6.13)$$

where $F(\xi, \eta, \zeta_W)$ is, by definition, the "wheel shape function."

Our objective will be to find $\zeta_W$ for given values of $\xi$ and $\eta$. However, we cannot solve Eq. (6.13) directly because $r_W$ appears in it implicitly in a highly nonlinear fashion. Therefore, we will use the Newton-Raphson iterative method [see Hamming, 1971, pp. 49-51] which requires that we guess at an initial value of $\zeta_W$ and then find an improved value

$$\zeta_W + \Delta \zeta_W \quad (6.14)$$

where the incremental correction is given by

$$\Delta \zeta_W = -\frac{F(\zeta_W)}{\frac{dF}{d\zeta_W}} \quad (6.15)$$

where $F$ is the wheel shape function defined by Eq. (6.13). Since $\zeta_W$ enters Eq. (6.13) through the terms $z_W$ and $\rho$, and $\rho$ is a function of $x$, we can evaluate $dF/d\zeta_W$ in the form

$$\frac{dF}{d\zeta_W} = \frac{\partial F}{\partial z} \frac{dz}{d\zeta} + \frac{\partial F}{\partial \rho} \frac{d\rho}{d\zeta} + \frac{\partial F}{\partial x} \frac{dx}{d\zeta} \quad (6.16)$$

The procedure is as follows:

a. Guess a value for $\zeta_W$ (for example, use Eq. (4.5) to find $\zeta_W$ corresponding to $\eta = 0$).
b. Find \(F(\zeta^W)\) from equation (6.13).

c. Find \(\frac{\partial x}{\partial \zeta}, \frac{\partial z}{\partial \zeta},\) and \(\frac{\partial \rho}{\partial x}\) from Eqs. (4.3) and (6.11), in the form

\[
\frac{\partial x}{\partial \zeta} = -\sin \theta_c \quad (6.17)
\]

\[
\frac{\partial z}{\partial \zeta} = \cos \theta_c \quad (6.18)
\]

\[
\frac{\partial \rho}{\partial x} = \left\{ \frac{\sqrt{R_1^2 - (x-a_i)^2}}{-(x-a_i)} \right\} \quad (arc)
\]

\[
\frac{\partial \rho}{\partial x} = -a_i \quad (straight) \quad (6.19)
\]

d. Find \(\rho\) from Eq. (6.11), then \(\frac{\partial F}{\partial \rho}\) by differentiating Eq. (6.13), i.e.

\[
\frac{\partial F}{\partial \rho} = -2\rho \quad (6.20)
\]

e. Find \(\frac{dF}{d\zeta}\) from Eq. (6.16)

f. Find \(\Delta \zeta\) from Eq. (6.15)

g. Find the new value for \(\zeta\) from:

\[
\zeta_{\text{new}} = \zeta_{\text{old}} + \Delta \zeta \quad (6.21)
\]

h. Repeat steps a through g, with the new value of \(\zeta\), until a desired tolerance is reached for \(|\Delta \zeta|\).

With \(\zeta^R\) and \(\zeta^W\) determined at any point of the \(\zeta-\eta\) plane, the separation is calculated from

\[
s = \zeta^W - \zeta^R \quad (6.22)
\]

Subroutine INSEP

The subroutine INSEP—which stands for "INITIAL SEParation"—has been based on the analysis just discussed. Its purpose is to supply the initial separation between rail and wheel (and the unit normal components only for program CONFORM),
to the calling program. It calls upon subprogram RAIL and MIDWEL to find \( \zeta^R \) and \( \zeta^W \) when \( \eta = 0 \); then INSEP calculates \( s = \zeta^W - \zeta^R \). For \( \eta \neq 0 \), \( \zeta^R \) remains the same and INSEP calls upon WHEEL to calculate \( \zeta^W \).

The organization of subroutine INSEP is shown in Fig. 8, where the various subprograms have the following purposes.*

- **Subroutine INSEP:** calculates \( s \), the initial separation between rail and wheel.
- **Subroutine RAIL:** calculates \( \zeta^R \) for any given \( \xi \) (also components of unit normal vector when used with CONFORM).
- **Subroutine MIDWEL:** calculates \( \zeta^W \) when \( \eta = 0 \).
- **Subroutine WHEEL:** calculates \( \zeta^W \) when \( \eta \neq 0 \).
- **Subroutine WHEELO:** calculates \( \zeta^W \) and \( d\zeta^W/dx \).

### 7. INITIAL CANDIDATE CONTACT BOUNDARY

An initial estimate of the contact patch is essential for the numerical solution of contact stress problems. The projection on the \( \xi-\zeta \) plane of the intersection curve of the two rigid surfaces, obtained by their interpenetration through a distance \( \delta \), along \( \zeta \)-axis is a good first approximation to the actual region of contact for a rigid body approach \( \delta \). This region is called the initial candidate contact patch and its boundary is called the interpenetration curve (see Fig. 9). The interpenetration curve is given by

\[
\zeta^W - \zeta^R = f(\xi, \eta) = \delta \tag{7.1}
\]

In general, it will not be possible to get an analytical expression for \( f(\xi, \eta) \). Therefore, we use \( \xi \) as the independent variable, and find the value of \( \eta(y) \) which satisfies Eq. (7.1) by means of the following procedure.

**Procedure to find interpenetration curve**

a. Calculate \( \zeta^R \) from Eq. (6.2) for a given \( \xi \).

b. Calculate \( \zeta^W \) from Eq. (7.1) as;

\[
\zeta^W = \delta + \zeta^R \tag{7.2}
\]

c. Calculate \( x \) from Eq. 4.3-a) using \( \xi \) and \( \zeta^W \).

*Note that the subprograms named RAIL, WHEEL and WHEELO used with Subroutine INSEP are different from similarly named subprograms used with program MIDSEP (Sec. 5). A complete FORTRAN listing of INSEP and all its associated subroutines is included with the listing of CONFORM in Paul and Hashemi [1978].
Fig. 8 Organization of subroutine INSEP

Fig. 9 Interpenetration of surfaces
d. Calculate \( z^W \) from Eq. (4.3-b) and \( \rho(x) \) from Eq. (6.11).

e. Calculate \( \eta(\xi) \) by solving (Eq. 6.10)

\[
y = \eta = \pm \sqrt{z^W R^W} \]  

(7.3)

With \( \eta \) calculated for any value of \( \xi \), the interpenetration curve is determined. Based on the above procedure, a FORTRAN program called INTERPEN was written to determine the interpenetration curve for any given rigid body displacement. For Step a, INTERPEN calls the same subroutine RAIL, as that called by program MIDSEP. For step e, INTERPEN calls a subroutine named YRW.

A complete User's Manual and sample problem for Program INTERPEN will be found in Appendix 3 of Paul and Hashemi [1978]. A FORTRAN listing of INTERPEN and its associated subroutines is also given in that reference.
REFERENCES


APPENDIX A. SMOOTH RAIL AND WHEEL PROFILE

It is desired to be able to write the algebraic equation of each segment of a wheel or rail profile from the information supplied on an engineering drawing. These drawings usually show circular arcs which are joined smoothly to straight lines.

Smooth transition from one segment to a following segment in a rail or wheel profile may occur in the following ways:

a. From a straight line segment to a circular arc segment;

b. From a circular arc segment to another circular arc segment;

c. From a circular arc segment to a straight segment.

In most cases the slope of straight segments, radius of a circular arc segment, and coordinates of segment end points are given.

Smooth Transition from a Straight Line to a Circular Arc Segment [Fig. A-1(a)]

If the slope of the line is \( a_{i-1} \), its \( y \)-intercept is \( b_{i-1} \). Let \( x_{2-1} \) be the greatest abscissa on segment \((i-1)\), also let \( R_i \) be the radius of the circular arc segment \( i \) with the end point abscissa \( x_i \) [see Fig. A-1(a)]. Then,

\[
AC = R_i \sin \theta = \frac{a_{i-1}}{\sqrt{1+a_{i-1}^2}} R_i \tag{A-1}
\]

\[
AB = R_i \cos \theta = \frac{1}{\sqrt{1+a_{i-1}^2}} R_i \tag{A-2}
\]

or

\[
a_i = x_{i-1} + \frac{a_{i-1}}{\sqrt{1+a_{i-1}^2}} R_i \tag{A-3}
\]

\[
b_i = z_{i-1} - \frac{R_i}{\sqrt{1+a_{i-1}^2}} \tag{A-4}
\]

where

\[
z_{i-1} = a_{i-1} x_{i-1} + b_{i-1} \tag{A-5}
\]
(a) Transition from straight line to circular arc

(b) Transition from circular arc to a different circular arc

(c) Transition from a circular arc to a straight line

Fig.A-1 Profile transitions
Smooth Transition from a Circular Arc Segment to Another Circular Arc Segment \[\text{[Fig. A-1(b)\]}\]

If we are given the arc center coordinates \((a_{i-1}, b_{i-1})\), the radius \(R_{i-1}\), the greatest end point coordinate \(x_{i-1}\) of segment \(i-1\), and \(R_i\) then:

radius of arc segment \(i\) is given, then:

\[
EC = (R_{i-1} - R_i) \sin \theta = (R_{i-1} - R_i) \frac{x_{i-1} - a_{i-1}}{R_{i-1}} \tag{A-6}
\]

\[
ED = (R_{i-1} - R_i) \cos \theta = (R_{i-1} - R_i) \left[ -\left( \frac{b_{i-1} - z_{i-1}}{R_{i-1}} \right) \right] \tag{A-7}
\]

\[
a_i = a_{i-1} + (1 - \frac{R_i}{R_{i-1}}) (x_{i-1} - a_{i-1}) \tag{A-8}
\]

\[
b_i = b_{i-1} - (1 - \frac{R_i}{R_{i-1}}) (b_{i-1} - z_{i-1}) \tag{A-9}
\]

where

\[
z_{i-1} = b_{i-1} + \left[ R_{i-1}^2 - (x_{i-1} - a_{i-1})^2 \right]^{1/2} \tag{A-10}
\]

Smooth Transition from a Circular Arc Segment to a Straight Line Segment \[\text{[Fig. A-1(c)\]}\]

If we are given the arc center coordinates \((a_{i-1}, b_{i-1})\), radius \(R_{i-1}\), and end point abscissa \(x_{i-1}\) of segment \(i-1\), then the slope of the straight segment is:

\[
a_i = -\tan \theta = -\frac{x_{i-1} - a_{i-1}}{z_{i-1} - b_{i-1}} \tag{A-11}
\]

and

\[
b_i = z_{i-1} - a_i x_{i-1} \tag{A-12}
\]

where

\[
z_{i-1} = b_{i-1} + \left[ R_{i-1}^2 - (x_{i-1} - a_{i-1})^2 \right]^{1/2} \tag{A-13}
\]
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